2329

OTS: 60-11,355

JPRS: 2329

25 March 1960

MATHEMATICAL LOGIC AND THE FOUNDATIONS OF MATHEMATICS

- USSR -

by S. A. Yanovskaya

Distributed by:

OFFICE OF TECHNICAL SERVICES
U. S. DEPARTMENT OF COMMERCE
WASHINGTON 25, D. C.
~~Price: $3.00~~

U. S. JOINT PUBLICATIONS RESEARCH SERVICE
205 EAST 42nd STREET, SUITE 300
NEW YORK 17, N. Y.

JPRS: 2329

CSO: 3054-N

MATHEMATICAL LOGIC AND THE FOUNDATIONS OF MATHEMATICS

/This is a translation of an article written by S. A. Yanovskaya[1] in Matematik v SSSR za Sorok Let (Mathematics in the USSR in Forty Years), Vol 1, Moscow, 1959, pp 13--120./

Introduction

The works of Soviet scientists on problems of mathematical logic and fundamentals of mathematics during 1917 -- 1947 were already covered in the collection "Mathematics in the USSR During Thirty Years." It is therefore necessary for us to dwell here only on a brief survey of the works performed during the past ten years. However, during the ten years elapsed from the time of the publication of the collection "Mathematics in the USSR in Thirty Years," the staffs of Soviet scientists, working creatively on problems of mathematical logic, have grown so much, that it is hardly possible to give in a brief survey article enough information concerning the works performed and the results obtained.

1. The work on mathematical logic is being carried out in this country in many scientific research seminars and institutes of the Academy of Sciences USSR, the universities, and the pedagogical institutes. The Mathematical Institute imeni V. A. Steklov of the Academy of Sciences, USSR has included since 1957 a special division for mathematical logic, headed by P. S. Novikov.

In Moscow and in Leningrad there have grown large schools of specialists on mathematical logic, including many of the students of P. S. Novikov, A. A. Markov, and

---

1. The article was written with the participation of S. I. Adyan, Z. I. Kozlova, A. V. Kuznetsov, A. A. Lyapunov, and V. A. Uspenskiy.

A. N. Kolmogorov. But the work on mathematical logic proceeds now in Riga, in Ivanovo, in Penza, in Gor'kiy, and in other cities of the Soviet Union. Participating more and more in the work on mathematical logic and its application are also representatives of other sciences: scientific workers in the field of technical sciences, linguists, philosophers, and others. Papers on mathematical logic became frequent at the sessions of the Moscow Mathematical Society and the Leningrad All-City Seminar, lectures on mathematical logic and associated disciplines are systematically delivered at many faculties of the Moscow and Leningrad Universities. At the Moscow University during recent years there have been carried out special seminars also on individual problems on mathematical logic, such as the theory of computable functions, algebraic logic and its multiple-valued generalizations, technical applications of mathematical logic, mathematical logic, and linguistics and many others. Special seminars on general-logical applications of mathematical logic have been systematically in operation at the Institute of Philosophy at the Academy of Sciences USSR, and on the philosophical faculty of the Moscow University. In the Moscow and Leningrad Universities, P. S. Novikov, A. A. Markov, and N. A. Shanin have read many new courses, as follows: on the fundamentals of mathematics (P. S. Novikov and A. A. Markov), on the theory of algorithms (A. A. Markov), on constructive mathematical logic (P. S. Novikov, A. A. Markov and N. A. Shanin), on constructive mathematical analysis (N. A. Shanin). (Certain results, first detailed in these courses, will be treated later on.) Many special courses have been delivered also by A. V. Kuznetsov, V. A. Uspenskiy, S. V. Yablonskiy, and S. A. Yanovskaya.

A great role in the matter of the development of mathematical logic in the USSR and exchange of experience in this scientific region with other countries have been played also by such factors as: the founding of the abstract journal Matematika /Mathematics/, which systematically reports on work in mathematical logic both in this country as well as abroad; the expansion of contacts with specialists on mathematical logic abroad, particularly in the countries of the Peoples' Democracies; the gathering of the Third All-Union

Mathematical Congress with a section on mathematical logic, in which approximately 50 papers were delivered. Of great importance was the publication of several original and translated books on mathematical logics and mathematics, including the basic work by A. A. Markov "Teoriya algorifmov" /Theory of Algorithms/ (1954), the monographs by P. S. Novikov "Ob algoritmicheskoy nerazreshimosti problemy tozhdestva slov v teorii grupp" /On the Algorithmic Insolubility of the Problem of Identity of Words in Theory of Groups/ (1955), and N. A. Shanin's "O nekotorykh logicheskikh problemakh arifmeticki" /On Certain Logical Problems of Arithmetic/ (1955), the translation of several known handbooks on mathematical logic and mathematics by D. Hilbert and W. Ackerman ("Fundamentals of Theoretical Logic," Moscow, Foreign Literature Press, 1947), by A. Tarski ("Introduction to Logic and Methodology of Deductive Sciences"), Moscow, Foreign Literature Press, 1948), S. C. Kleene ("Introduction of Meta-mathematics," Moscow, Foreign Literature Press, 1957), and the translation of the monograph by R. Peter ("Recursive Functions," Moscow, Foreign Literature Press, 1954). As a rule, the translations are provided in our country with remarks, commentaries, supplements, and forewords, which reflect the work performed on the book by the editor and the translator. Thus, the foreword to the book by R. Peter was written by the editor of the translation, A. N. Kolmogorov.

2. Heretofore, the basic interests of the Soviet specialists on mathematical logic are concentrated around the difficult problems of mathematics and its foundations; our specialists are glad to engage also in problems of mathematical logic, which arise in other sciences, above all in connection with the construction of automatic machines, including complicated information machines, which propose the development of various problems not only in engineering, but also in mathematical linguistics.

In order to solve a problem to prove a theorem, to find a general method (algorithm) for an effective solution of an entire class of homogeneous problems, to reduce the solution of one problem to the solution of another (or others), the mathematician does not have usually to think much on what

in general is meant by "solve a problem" "prove a theorem," what is a "mass problem" or the "algorithm" that solves it, what is meant by "reduction of one problem to another (or others)?" But all these problems occur unavoidably when the problem refuses to be solved stubbornly, when the theorem can neither be proved nor disproved, when the algorithm cannot be found. It is in these problems that mathematical logic engages and particularly its principal branch, meta-mathematics, a science whose object are mathematical theories and problems. But in order for us to be able to use the solution of these problems in mathematics, meta-mathematics must be constructive: it should above all bear itself an exact mathematical character. As has been shown in the development of modern mathematical logic, such a statement of the problem presupposes a special study of constructive objects and methods of mathematics, their role and significances in all of mathematics, including the mathematics that operates with non-constructive objects or using such (non-constructive) means as the use of law of excluded third in discussions of multiple sets. These types of problems are reflected above all in the theory of algorithms (absolute and so-called reducibility algorithms, which solve each of a certain class of homogeneous problems only under the condition that there exists additional information, the obtaining of which no longer has, generally speaking, an slogirthmic character).

Also belonging to the theory of algorithms of both kinds and to the theory of computable (recursive) functions and operators, the generalizations of which permit us to cope already also with the structure of non-constructive objects of mathematics, and enable us to give a classification for broad classes of sets, functions, predicates, and other objects of mathematics and meta-mathematics, to clarify the limits of capabilities of their constructivization, their role in the solution of problems of mathematics, and its justifications. The contribution of Soviet mathematicians to this circle of problems is treated in the central (second) chapter of our survey, devoted to the theory of algorithms and computable functions and operators.

The differences in the points of view on the problem

of the meaning of non-constructive objects and methods of mathematics of the "Moscow" School of students and successors of P. S. Novikov and A. N. Kolmogorov and the "Leningrad " School of students of A. A. Markov expresses itself already in this chapter of the survey. The authors have tried to expound as objectively as possible material pertaining to problems still under discussion. Naturally, however, the point of view which they consider the broadest and most convincing and which consists of including in their work not only constructive but also classical methods in mathematics and mathematical logic, has found its reflection in the formulation of this article and in the treatment of the material.

3. The third chapter of the survey is devoted to applications of the theory of algorithms both to proofs of the insolubility of several central algorithmic problems of algebra and topology (the corresponding section 9 was written in its entirety by S. I. Adyan), and to problems of constructivization of mathematics, particularly of mathematical analysis, which are considered in themselves both from the constructive (after A. A. Markov) and from the classical point of view.

The object of the fourth chapter is logical and logical-mathematical calculations, which are considered also above all in light of theory of algoriths. A particular place is occupied in this chapter by the section devoted to the algebraic logic (Section 13, written by A. V. Kuznetsov). The origin of the problematics discussed in this section is connected to a considerable extent with problems which arise in engineering, with problems of analysis and synthesis of relay systems and other automatic devices, with the problem of the minimization of the number of contacts in the circuit, with the problem of programming, etc. It was intended initially to include in this survey the entire general circle of problems, connected with technical applications of mathematical logic. However, since the corresponding literature is treated in the survey of papers on cybernetics, we have resolved to forego its discussion in the present survey. We note merely that from the point of view of connection with mathematical logic one must consider above all the

papers by M. A. Gavrilov, A. V. Kuznetsov, O. B. Lupanov, A. G. Lunts, A. A. Markov, G. N. Povarov, V. N. Roginskiy, B. N. Trakhtenbrot. B. I. Shestakov, and S. V. Yablonskiy, on which we shall dwell partially in Section 13, devoted to algebraic logic and its multiple-valued generalizations.

From among the other applications of mathematical logic, a particular place is occupied by applications to linguistics. On these, and also on certain other problems, which are not reflected in the principal part of the survey, we shall dwell in the small concluding section.

4. Problems traditionally included in courses on the foundations of mathematics and which essentially do not presuppose as yet the theory of algorithms, have been relegated by us to the first section of the survey. Here we report on papers by Soviet authors, devoted to axiomatic theory of sets (proofs of non-contradiction of certain hypothesis of descriptive set theory; proof of the incompleteness of a wide mass of axiomatic theories of set, based on the existence in them of unprovable means of premises concerning the equivalents of definitions of a finite set; problem of antinomies of set theory, etc.).

The second section, written by A. A. Lyapunov and Z. I. Kozlova represents an independent survey, devoted to problems on descriptive theory of sets. Its inclusion in the survey on work on mathematical logic and fundamentals of mathematics is explained by the fact that in the works of the representatives of the Moscow school of mathematical logic, the methods of descriptive theory ofsets play a particular role. The analogies of descriptive classifications of sets and functions are used in classification of "arithmetic" sets, functions, predicates, transfinites, and other objects (Section 8, written with participation of A. V. Kuznetsov). Topological properties of Baire space make it possible to cope with the structure of constructive mathematical analysis and constructive logic (Sections 10 and 12). They play a substantial role in the solution of difficult problems, pertaining to the problems of reducibility. (Sections 6 and 7).

5. We already mentioned the discussions between the representative of the constructive and classical trends among

the Soviet specialists in mathematical logic. One must note however, that these arguments are carried out against a common background of dialectic materialism, so that both the "constructivists" and the "classicists" are decisively fighting against idealistic misinterpretations of mathematical logic. For illustrations, we give here two examples. The first is borrowed from the foreword by A. N. Komogorov to the book by R. Peter "Recursive Functions." A. N. Kolmogorov comes out here against the agnostic "deductions" from proofs of insolubility of a certain mass (algorithmic) problem. The essence of the matter lies even not only in the fact that one proves this way only the non-existence of a single method, with which one could solve any of a certain infinite class of problems, but above all in that such proofs have positive cognitive meaning. Thus, "it is easy to show that from the algorithmic unsolvability of the Fermat problem in the sense of R. Peter there follows a negative solution of the general Fermat problem" (p. 9). Here A. N. Kolmogorov emphasizes that although the constructive trend in mathematics makes wide use of the constructive results obtained in the "intuitionistic" school founded by Brouwer, but actually the positive accomplishments of the constructive trend have no relation whatever to the philosophy of intuitionism." He therefore considers it necessary to note particularly that the "use of intuitionism or the terms "intuitionistic logic" and "intuitionistic arithmetic" as "technical terms" on the part of many authors who are far from the philosophy of intuitionism, leads to great confusion and should be recognized as being false (p. 9).

N. A. Shanin, the logical-mathematical research of whom are devoted to the development of problems of the constructive trend in mathematics, also writes on the same subject. In his paper "On Certain Logical Problems of Arithmetic" we read: "The constructive trend in mathematics began to take form at the beginning of the present century and at the initial stages of its development it was connected with the philosophical current in mathematics, called intuitionism. One must emphasize that the methodological premises of intuitionism are highly inconsistent. It is enough to indicate for example that the founders of intuitionism,

Brouwer and H. Weyle, consider the concept of natural number not as a result of the abstracting work of human thought, which processes the rich social experience of operation with various groups of objects, but as a manifestation of "initial intuition." However, further progress in science has proved convincingly that the real contents of constructive trend in mathematics is no way connected with the methodological premises of intuitionism, and is caused by specific mathematical problems of a special type, the investigation of which is of considerable interest both for mathematics itself, as well as for its applications. The development of mathematics has exhibited the need for all out investigation of various processes of construction and of potentially realizable results of the development of such processes. It is indeed the problems of this character that make up the scope and methods of the constructive trend in mathematics" (p. 4).

6. Inasmuch as the present survey is primarily of summary character, the authors naturally did not pretend to a sufficiently complete exposition of the contents of the papers reviewed. We attempted merely to give the reader a certain -- albeit rought -- idea on the problems studied in them and on the results obtained. Most frequently we used in this case the terminology and symbols of the author of the paper. If sufficiently widely used terms or symbols were referred to, we did not stop to explain them. In many cases we did not warn against the incomplete rigor of our formulations.

Many results elucidated in the present paper were reported and proved in all the details only at the sessions of the Scientific-Research seminars, the minutes of which we had to use consequently. At the Moscow University such minutes are kept in detail and reliably since Fall of 1946 by the Secretary of the Seminar on Mathematical Logic. A. V. Kuznetsov, whose materials were extensively used in the present survey. Unfortunately, the authors did not always have at their disposal the corresponding materials on other seminars, particularly on the seminar of the Leningrad Division of the Mathematical Institute of the Academy of

Sciences USSR.[1]

For consultation and aid in writing this survey, I consider it my duty to thank particularly A. V. Kuznetsov and V. A. Uspenskiy.

1. Note added in proof. A collection of works of the Leningrad Seminar was published in 1958 in the 52nd volume of the works of the Mathematical Institute imeni V. A. Steklov.

Chapter I

CERTAIN PROBLEMS OF SET THEORY

1. Axiomatic Set Theories

1. As is known, the difficulty connected with the basics of the set theory cannot be overcome with the aid of formulating it in the form of a sufficiently deductive form of theory, based on the choice of a certain system of axioms and finite rules of deduction. Even if one disregards the principal incompleteness of these theories (the "Goedel Theorem") the non-contradiction of such a theory T_1 -- by virtue of the second Goedel theorem -- cannot be proved by means of this theory itself (the T_1 theory); such a proof presupposes the existence of another theory of sets T_2, which is sufficiently strong to be able to carry out in it the proof of the non-contradiction of the first theorem, which at the same time is non-contradictive. Thus, it is difficult to count on a proof of non-contradiction for any axiomatic set theory. It is also known that for all existing axiomatic systems of set theory, the rules of deduction of which are formalized in the narrow calculus of predicates and in which one can prove the existence of non-denumerable sets, we come up against the so-called Skolem paradox, according to which there follows from the assumption of the feasibility of such a system also its feasibility in the denumerable model (i.e., the image of a set, which is non-denumerable inside a given axiomatic system, is found to be a certain subset of a denumerable model of this system). It is therefore natural to consider proofs of non-denumerability, carried out within elementary (i.e., formalized by means of narrow calculus of predicates) of axiomatic systems of set theory, only as proofs of the relative non-denumerability, leaving the problem of the existence of absolutely non-denumerable sets open.

Nevertheless, difficulties of this kind do not deprive the axiomatic constructions of set theory of scientific interest. It is enough to note that with their aid it is possible to throw light on many difficult problems in the theory of sets by proving relative non-contradiction, such,

for example, as the known theorems of Goedel on the non-contradiction of the continuum-hypothesis and axioms of selection.

From the works of Soviet mathematicians, it is necessary to note in this connection above all the results of P. S. Novikov [25], pertaining to the following principal problems of descriptive theory of sets:

1) Problem of the cardinality of a complement to the A-set.

2) Problems of measurability of projections of complements to an A-set.

3) Problem of separability of the projective sets of higher classes.

The first two of these problems were posed already in 1930 by N. N. Luzin [90] (see also P. S. Novikov [25], p. 279) who expressed concerning them an opinion that they are of the same nature as the problems of the continuum, i.e., that their difficulty is due not to a shortage of means of construction or derivation at the disposal of the mathematicians, but the fact there are not enough generally-accepted premises of set theory for the deduction of an answer of these problems from these.

But in order that judgment of this kind acquire an exact meaning, it is necessary to know what is really meant by "generally accepted premises of set theory," and to indicate methods that permit, in spite of the limited possibility, to deduct consequences from these premises, and to become convinced that among these consequences one cannot find an answer to the raised questions. In other words, it is necessary to resort to a "formalized" axiomatic construction of theory of sets, if possible a stronger one. Such are the well known system of Goedel, which is sufficiently strong to enable one to realize in it all the conclusions that are usually obtained in the existing set theory. By means of his system (called by him the system) Goedel showed that the generalized continuum hypothesis

$$2^{\aleph_\alpha} = \aleph_{\alpha+1}$$

does not contradict the system of the axioms of the theory of sets (system Σ), if this system itself is not contradictory. In another paper (see Symbolic Logic, Vol. 116) Goedel published without proof the following statement:

If a system of Neumann axioms for set theory is non-contradictory, then it remains non-contradictory also after joining to it two new axioms, which establish the existence of such complements to the analytic sets, which have the cardinality of a continuum, but do not contain a perfect subset, and such linear B_2 sets, which are not measurable in the sense of Lebesgue.

The proof of this statement still does not resolve completely the question raised by N. N. Luzin concerning non-derivability. However, it represents a considerable step forward towards the solution of the problem and in general, towards foundation of set theory. P. S. Novikov [25], was first to publish the proof of the non-contradiction (in the Σ system) of both premises:

a) Concerning the existence of a non-denumerable complement to the A-set, containing no perfect subset.[1]

b) The existence of a set of type B_2, which is not measurable in the sense of Lebesgue.[2]

(The proof of K. Goedel has thus far not been published.

In addition to that, P. S. Novikov [25] established non-contradiction (in the Σ system of axioms of set theory) of the following statement concerning the laws of separability of projective sets of higher classes: starting with a certain n, the laws of separability for

---

1. "The exceeding importance of this problem is caused by the fact that if its solution is negative, then the famous problem of continuum is solved affirmatively"(N. N. Luzin, [90], p. 288).

2. "The author considers as unresolved the problem of whether all the projective sets are measurable (in the sense of Lebesgue) or not" (N. N. Luzin [90], p. 321). After the proved non-contradiction of the statement concerning the existence, as proved by P. S. Novikov, of a projective set which is not measurable in the sense of Lebesgue, it becomes necessary, in order to prove the non-solvability (in the Σ system) of the problem raised by N. N. Luzin, to prove also the non-contradiction (in the Σ system) of the system that all the projective sets are measurable in the sense of Lebesgue.

sets A_n (projective sets of class n) is the same as in the second class, but not the same as in the first class of projective sets). (The inverse character of the laws of separability for projective sets of second class relative to those established by N. N. Luzin [90], p. 155, 206 to the laws of separability for the class of A sets was observed by P. S. Novikov [8, 10] in 1935. More accurate formulations can be found in [25], p. 280, and in the remarks of [90], pp. 355 -- 356.)

The proofs of P. S. Novikov are based on the fact that in the Σ system it is possible to separate a Baire space J (respectively J^n) the points of which correspond to real numbers (respectively aggregates of n real numbers), and to develop the ordinary descriptive theory of sets in the Baire space. These proofs have the geometric character that is characteristic of the Luzin school, and consists of observing that in the Δ model of the Σ system, in which all sets are constructive in a definite sense, statements (a) and (b) are true. If the Σ system is non-contradictory, then part of its Δ is also non-contradictory, and therefore the statements (a) and (b) which are proved in Σ are also non-contradictory in Δ (and also the statement (c): concerning the existence of the function defined in all points of Baire space, belonging to class A_2 and discontinuous on each perfect set). Premise (c) -- the non-refutability of which follows from its provability in the Δ model, is also of independent interest -- is used to prove premises (a) and (b) in the Δ model. The proof of the results, pertaining to problems of separability of projective sets of higher classes, has as yet not been published, although it has been discussed in all its details by P. S. Novikov at many sessions of the Seminar on Mathematical Logic at the Moscow University as early as 1950.

2. The ideas and methods of P. S. Novikov were extensively used and developed in the work of his students and participants of the Seminar on Mathematical Logic at the Moscow University. We shall encounter these methods in Section 10, where we speak of constructive mathematical analysis, and also in Sections 6 and 7, devoted to the algorithms of reducibility or corresponding problems of computable operators. We shall also encounter

them in Section 12, when speaking of the connection of the Goedel theorem concerning the incompleteness (and non-complementarity) of the formalized arithmetic (and also theorems on the non-solvability of problems of solvability on finite classes) with problems of non-separability, and in many other cases. Here we shall dwell briefly only on the results of B. S. Sodnomov [4, 7] who directly continued the problematics of the paper by P. S. Novikov [25]

Using the trigonometric methods of P. S. Novikov, B. S. Sodnomov [4] has shown that if a system Σ is non-contradictory also after joining to it the axiom which states that if (S) is a family of subsets in the interval (0, 1) such that there exists for it a universal projective set A of class a, then among the sets constructed by applying the axiom of selection to the system $\{S\}$, there exists a projective set of class not higher than $(2, a) + 2$.[1] By way of consequences of this theory, B. S. Sodnomov obtains proofs of the non-contradiction (relative to the Σ system) of the following statements:

(a) Among the sets that are not measurable in the sense of Lebesgue, obtained by a choice of one point each in one system of rationalities (the Van der Warden example), there exists a projective set of class not higher than third.

(b) The well-known Hausdorf breakdown of a sphere can be realized and projective sets of not higher than third class.

Later on (see [7]) B. S. Sodnomov has increased considerably the number of such results, pertaining to the non-contradiction of projectivity of certain remarkable sets and to an upper limit of their classes, leaning here on a study of the role of the operation of arithmetic addition (term by term addition) of two sets of real numbers and formation of sets that are not measurable in the sense of Lebesgue.

---

1. An analogous result -- without an exact estimate of the class of projectivity -- was obtained earlier by Kuratowski by mathematical-logic methods (see C. Kuratowski, Ensembles projectifs et ensembles singulaires, Fund. Math. 35, (1948), 1931-140).

3. A series of results on the axiomatics of the theory of sets was obtained by A. S. Yesenin-Vol'pin. He considers a system of axioms Θ, the principal concepts of which are: object, class, relation of belonging, which can take place only between the object and class. The classes that are objects are called sets.

Axioms: analogous to the axioms of the Σ system of the work of Goedel on the continuum hypothesis: axiom A1 of Goedel, that any set is a class, is replaced by an axiom that the element is always an object.

Results:

1) Non-contradiction of the statement of the existence of an infinite set, between the cardinality of which and the cardinality of the set of its subsets there are intermediate cardinalities (this was proved on 29 June 1951 and published in 1956 [1.]).

2) The non-provability of the Suslin hypothesis in the system Θ [3]. (The Suslin hypothesis is that in a continuously ordered set, any system of paired non-intersecting intervals of which is not more than denumerable, must have a denumerable dense subset).

It is important to emphasize that in both these cases one deals with non-provability in a system of axioms, which does not contain a selection axiom. The proof consists of constructing a model, in which there is a set x such that between the cardinalities of the sets x and P (x) (the set of all subsets of set x) there is a set of intermediate cardinality, and also a model in which there takes place the negation of the Suslin hypothesis; on the other, the selection axiom does not take place in these models, thus making this axiom independent.

The latter result -- the independence of the selection axiom -- was obtained by Mostowski as early as in 1938 or 1939. The work of A. S. Yesenin-Vol'pin[1.] was carried out independently of the work of Mostowski and does not contain the additional theorem (published in 1954 [3]) that if a class satisfies certain five conditions, it can be used for the construction of a model; the Goedel model (Δ model), that of Mostowski, and that of Yesenin-Vol'pin are constructed by the latter on the

---

1. A. S. Yesenin-Vol'pin. Review of the book by Bachmann "Transfinite Numbers. Novyye knigi za rubezhom [New Books Abroad] 7, 1956

basis of this theorem, the proof of which is given by the Goedel method. This theorem has made it possible for the Hungarian mathematician Hajnal to prove that if the hypothesis $2^{\aleph_\alpha} = \aleph_{\alpha+\beta+\gamma+1}$ is non-contradictory, then the hypothesis $2^{\aleph_\alpha} = \aleph_{\alpha+\beta+1} \& (\delta)(\delta > \alpha+\beta+\gamma \supset 2^{\aleph_\delta} = \aleph_{\delta+1}$

is also non-contradictory, so that, for example, if $2^{\aleph_0} = \aleph_3$ is non-contradictory, then $2^{\aleph_0} = \aleph_2$ is also non contradictory, and from the provability of $2^{\aleph_0} \neq \aleph_2$ in the Σ system with the selection axiom then follows the probability of the continuum hypothesis in the same system (Andras Hajnal, Acta Scientarum).

These results are true also for the system "A, B, C" which results from the $_1$ system by removing the "consolidation" axiom "D".[1] As regards the latter, A. S. Yesenin-Vol'pin has proved that it is possible in its formulation to use instead of the quantor by classes an analogous quantor by sets ([3], footnote on p. 10), so that, for example, in the Zermello-Frenkel system it is possible to formulate the consolidation axiom in the form of a single axiom, and not a scheme of axioms, as was done by Mostowski (Fund. Math. 1950, 111 -- 124).

In 1951 A. S. Yesenin-Vol'pin investigated also the problem of what models for set theory can be constructed by means of the Θ system (or Σ system -- in [3] the model for Θ with infinite set of objects-nonclasses was constructed by means of the Σ system). In the same year Sheperdson (England) published the first part of an analogous investigation, while the second and third part appeared in 1952 and 1953. Among the other results of Sheperdson he shows that by means of the system it is impossible to construct a "standard model" for the Σ system, in which the Goedel axiom of constructivity would be violated (from which, in particular, there follows the selection axiom and $2^{\aleph_\alpha} = \aleph_{\alpha+1}$); the term "standard model" denotes here a model, forms if it is considered in the initial system (with which means one constructs this model), a

1. The result of the independence of the selection axiom in the system "A, B, C" was also obtained by Mendelson (1955) and Specker (1957) Zeitschr. math. Logik u. Grundl. Math. 3, (1957)).

class that is fully ordered in type of a certain ordinal (i.e., the ordinal number of class of all ordinal numbers). Sheperdson considers the "A, B, C," system described above. Independent of him, A. S. Yesenin-Vol'pin obtained an analogous result for the Θ system, to which an axiom is added concerning the complete ordering of the class of all individuums (i.e., objective-nonclasses). A particular case of such a system as the "A, B, C" system. (This result was formulated by A. S. Yesenin-Vol'pin at the end of his survey.[1])

From among the separate results pertaining to the Goedel system we note also the dependence observed by A. A. Markov [37] of one of the axioms of this system, namely the axiom B6 (which states, that for any class A there exists a class B, in which the ordered pair enters if and only if the pair $\langle yx \rangle$ enters in A), on the remaining axioms of this system (namely axioms A4, B5, B8, B4).

4. One of the principal obstacles along the path of an axiomatic basis of set theory is the deductive incompleteness (and incompletability) of all ordinary (elementary) axiomatic systems of set theory, discovered by Goedel and formulated in his famous theorem (reinforced by rosser and others).[2] Concerning the connection of this incompleteness with the effective non-separability of the set of proved from the set of refuted formulas of such theories we shall speak later (in connection with the works of A. V. Uspenskiy and B. A. Trakhtenbrot) in Section 12. Here we note only that the incompleteness proof, based on the concept of recursive non-separability, proposed by B. A. Trakhtenbrot[3] [1,11] makes it possible to

---

1. A. S. Yesenin Vol'pin, Review of the article by Shepherdson "Internal Models for Set Theory III" (Journal of Symbolic Logic, 18:2 (1953), 145 -- 167), published in Referat Zhur Matematika 3, 1954), Abstract No. 2491.
2. A more accurate formulation of the Goedel theorem can be seen, for example, in the article by B. A. Trakhtenbrot [2], Section 42.
3. B. A. Trakhtenbrot. The problem of solvability of finite classes a definition of a finite set. Author's abstract of dissertation, Kiev, 1950.

exhibit the existence, in all (elementary) axiomatic set theories, of unsolvable premises concerning the equivalence of two definitions of finiteness of a set. Since this gave the final answer to the long-posed question of theory of finite sets, we shall dwell on this result by B. A. Trakhtenbrot in somewhat greater detail.

The best known of all the definitions of finiteness (or respectively infiniteness) of a set is the definition given by Dedekind for a finite set as not equivalent to any of its regular subsets. Other authors (Zermelo, Russell and Whitehead, Tarski, Neumann, Weber-Stoekel, and others) proposed a series of different definitions (thus, for example, the definition of Weber-Stoekel says: "The set is finite if it can be ordered in such a way, that any subset in it has, in the same order both the first, and the last element"). The naturally-arising question of whether a set, which is finite in the sense of one of the definitions, also finite in the set of some other definition, it was found in many cases that within the framework of the given axiomatic theory it is impossible to answer this question: is it necessary to extend this theory by joining to it certain new asioms? Thus, in his "Ten Lectures on Principles of Set Theory," A. Frenkel indicated that on the basis of the system of axioms proposed by him for set theory, a system not including the free-choice axiom, it was impossible to prove the equivalents of the Dedekind definition to some other definition; but he stated in the same place the opinion that with the selection axiom it is possible to prove already the equivalents of any two definitions of a finite set.

In considering the wide class of logical-mathematical calculi $\mathcal{T}$, obtained by adding to the narrow calculus of predicates (with identity axioms) a finite (or, more generally a recursive-denumerable) system of axioms (the Σ system of Goedel-Bernais axioms, which was already mentioned above, like any of its non-contradictive expansion through addition of new axioms, belongs among the number of such calculations), B. A. Trakhtenbrot [11] has shown that Frenkel's hypothesis (as applied to all these calculations) is incorrect. Thus it is clear that by joining, for example, the selection axiom to the calculations of Goedel-Bernais (and even any denumerable set

of axioms) it is possible to obtain a calculus in which one can prove the equivalence of any two definitions of a finite set (i.e., of any two formulas of the τ calculus, which are identically true in all the finite regions).

Furthermore, B. A. Trakhtenbrot has proved also that ([11], theorem 4) in the axiomatic set theory there exists no strongest or weakest definition of a finite set. (The definition A is considered stronger, more accurately no less weaker than B, if in it is shown the considered calculus that any set, finite in the sense of definition A, is finite in the sense of definition B). An analogous result for a sufficiently broad class of definitions of a finite set was obtained in 1938 by A. Mostowski;[1] however, it is given in the article without proof. The premises stated by Mostowski in this article are readily proved on the basis of theorems established by B. A. Trakhtenbrot in his articles [1, 2].

5. The difficulties connected with basing a set theory are caused to a considerable extent by the fact that one transfers to infinite sets the methods of operation with finite sets, the elements of which are considered in this case as absolutely "solid" bodies, each of which is fully distinguishable from any other and continues to remain "itself," being included in any set of objects or their sets. If the mathematical objects do not satisfy this kind of requirement of "accuracy," they are usually excluded from consideration, and the mathematical concepts are defined in such a way, that only objects of this nature satisfy them. The antinomy (paradoxes) of so-called naive theory of sets can be considered as evidence of the fact that "accuracy" requirements of this kind (and "discreteness") of objects are not observed in this theory. Various types of axiomatic theories of sets take it upon themselves to eliminate this effect. To what extent are they successful in it, however? Thus, in particular, to

1. A. Mostowski, Uber den Begriff der endlichen Menge. Sprawozdania Tow. rank. Warsz. Wydzial III (1938), 13. The principal possibility of the existence of a denumerable set of deductively independent formulas, identical with the finite ones, was proved by Wajsberg in 1933 (see M. Wajsberg, Untersuchungen über den Funtionenkalkül für endliche Individuen berieche, Math. Ann. 108 (1933), 218 -- 228).

prove the non-contradiction of such theories, bearing in mind Goedel's second theorem, according to which the non-contradiction of the sufficiently strong (formalized) theory cannot be proved by means formalized in the same theory? One of the attempts of proving the non-contradiction of the axiomatic theory of sets by means of another theory, which is known not to satisfy the requirement of "accuracy" of its concepts and therefore cannot for formalized in the form of such a system, to which the second Goedel theorem is applicable, was undertaken by A. S. Yesenin-Vol'pin.

A. S. Yesenin-Vol'pin[1] criticizes the abstract potential realizability, which he considers to be merely an idealization. In particular, he raises doubts concerning the reliability of the principle of mathematical induction $(A(0) \& (n)(A(n) \supset A(n+1))) \supset (n)A(n)$, inasmuch as "in order to extract $A(10^{21})$ из $A(0) \& (n)(A(n) \supset A(n+1))$, without resorting to this principle, it is necessary to go through 10^{21} steps, which is impossible." In this connection, A. S. Yesenin-Vol'pin introduces the concept of "realizability" as denoting the actual, and not idealized realizability. From this point of view it is possible to assume that 0 is a realizable number and if n is realizable, then n + 1 is realizable -- and still assume that there exist non-realizable numbers, inasmuch, without resorting to the principle of mathematical induction, it is impossible to derive a contradiction from this. The principle of complete mathematical induction is found to be true for realizable numbers (under the assumption that the operation of writing out formula A(n) and a transition from A(n) to A(n + 1) for a realizable n is realizable). One can speak also of a "relative realizability," i.e., realizability of numbers or other mathematical objects under the assumption that certain definite numbers or objects and finite-value functions are considered to be realizable; in this case the reserve of relatively realizable objects is determined by the possibilities, which are available to us under conditions of the given problem, of constructing certain objects from other objects, so

1. A. S. Yesenin-Vol'pin, Analysis of Potential Realizability. In press (first reported in 1956).

that there are objects which are non-realizable in the given relative sense. This point of view is called by A. S. Yesenin-Vol'pin "outspoken," in contrast with the "traditional," which acknowledges only one, idealized, realizability.

The concepts of the "outspoken" point of view appear dim in the traditional point of view, and therefore cannot be formalized in the "traditional" systems (although there is a possibility of considering "outspoken formalisms"). This causes the author to hope that the second Goedel theorem is no longer an obstacle to prove the non-contradiction of a classical theory of sets by means of his "outspoken" theory. Using the idea of the relative character of realizability, the author replaces the concept of infinity by a concept of non-realizability and obtains in this way, convincing from his point of view, proof of non-contradiction of set theory, namely a system of Zermelo axioms. Using on the other hand successively stronger and stronger hypothesis concerning the realizability of functions, defined purposefully with the aid of "outspoken" concepts, he also justifies the non-contradiction of system with further axioms on the existence of "non-attainable cardinal numbers" (i.e., non-denumerable alephs with limiting indices, which are not limits of a smaller number of smaller alephs).

6. Problems concerning antinomies (paradoxes)[1] and the associated "paradoxial" consequences[2] -- the "naive" theory of sets and the corresponding broadened calculus of predicates without the theory of types have engaged, during the time of interest to us, the attention of P. S. Novikov and D. A. Bochvar. In one of his earlier papers, D. A. Bochvar [4] separated, as is known, in the broadened predicate calculus without type theory the known non-contradictory part (K_0 calculus), which contains no individual predicate symbols. (Any individual predicate -- including also

1. The word "paradox" will denote from now on a contradictory expression (antinomy).
2. There exists a general method of exclusion (elimination) in paradox from a deductive system of the type of broadened predicate calculus. It is enough to note that in the

one defined only in terms of logical constants (negation, conjunction, disjunction, implication, quantors, and equality) are classed by D. A. Bochvar already to non-logical constants, thus separating mathematics from logic in systems of the type Principia Mathematica of Russell and Whitehead). In the K_0 calculus, therefore, there is no "convolution" axiom[1], which permits the introduction of individual predicates by means of definitions of the type

$$p(\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_n) \underset{Df}{\equiv} \mathfrak{A}(\mathfrak{v}_1, \mathfrak{v}_2, \ldots, \mathfrak{v}_n)$$

(where p is the individual predicate of symbol and $\mathfrak{A}(\mathfrak{v}_1, \ldots, \mathfrak{v}_n)$ -- a formula containing free variables $\mathfrak{v}_1, \ldots, \mathfrak{v}_n$), since in general there are no rules from which there would follow a statement of non-trival connections of existentional character between the object and the predicate.

In the construction of a formalized (axiomatic) theory of sets (generally, any kind of logical-mathematical system) it is necessary to add, however, to the axioms and rules of logic precisely statements of existentional

---

Footnote 2 cont. from p. 21... ...derivation of the paradox $\mathfrak{A} \equiv \neg\mathfrak{A}$ we used some sort of assumption $\mathfrak{B}$, which is not deriavable in a purely logical (non-contradicting) part of the system ("absolute logical calculus" K_0 of D. A. Bochvar [6]), -- for example, using the axiom of spatiality, the axiom of selection, the axiom of "convolution," or still others of this kind, in order, by introducing in explicit form a reference to the used assumption $\mathfrak{B}$, to obtain instead of the paradox $\mathfrak{A} \equiv \neg\mathfrak{A}$, for example, the formula $\mathfrak{A} \equiv (\neg\mathfrak{A}) \& \mathfrak{B}$, which in itself is no longer a paradox: from it it follows simply that both $\mathfrak{A}$ and $\mathfrak{B}$ are false. The formula $\neg\mathfrak{B}$ proved in this manner is called, after P. S. Novikov, "a paradoxical consequence" (see [18], pp. 22 and 23). In the general case such a "proof" can hardly be considered, however, as a convincing refutation of the assumption $\mathfrak{B}$; too much could be"proved" in this manner.

1. The axiom "convolution" is sometimes called the abstraction principle, since it permits the introduction of new concepts. According to D. A. Bochvar, such a definition always contains in itself an existence statement.

character. Does not this lead to a contradiction?

In reference [18] P. S. Novikov indicated the type of axioms that could be joined without contradiction to K_0 in any number. Namely, it was found (theorem 1) that in order that a deductive system, formed by adding to K_0 axioms of the type

$$(Ep)(x_1)\ldots(x_n)[p(x_1, \ldots, x_n) \equiv G(x_1, \ldots, x_n)] \qquad (1)$$

(corresponding to a statement of applicability of the rule of convolution to formula $G(x_1,\ldots,x_n)$) to be non-contradictory, it is sufficient that each variable in formula (1) occupy either only an internal or else only an external place (one says that a variable (given its inclusion) occupies an internal place, if it is under the sign of elementary predicate, and in the opposite case one says that it occupies an external place). The logical system formed by joining to K_0 axioms of the indicated type, is called by P. S. Novikov a system of type (T).

Considering further the paradoxical consequences[1] of a definite type from the definitions $p(x) \equiv G(x)$, where p is the sign of single-place individual predicate, P. S. Novikov ([18], theorem 2) establishes (for the case when in the formula G(x) each connected variable occupies only either an internal or only an external place), that if the paradoxical consequences of such definitions are derivable in some system of type (T), then the joining of the formula $p(x) \equiv G(x)$ to any system of type (T) does not lead to a contradiction.

It follows from theorems 1 and 2 that definitions of such predicates, which are constants, for example $p(x) \equiv 1$ (where 1 is the sign of truth), definitions of identity, reflexivity, transitivity, and also definitions of integers, given in Principia Mathematica, being joined to the system K_0, do not lead to contradiction.

The concluding part of the remark by P. S. Novikov

1. In the K_0 calculus (where any definition of an individual predicate is considered as an axiom, stating its existence), the paradoxical consequences of the definition of the predicate lose their paradoxical character: they become simply conditions for non-contradiction of a given predicate.

[18] is devoted to the problem of the general description of any contradictory (single-place) predicates. Formulas are given, which are necessary and sufficient conditions for a predicate p(x) to be contradictory. These conditions permit also to find effectively the contradictory predicates. Furthermore, it is found that with their aid it is possible to outline (with accuracy up to equivalence in K_0[1].) the general type of any contradictory (single-place) predicate.

In reference [6] D. A. Bochvar considers certain methods of adding to the K_0 calculus of individual predicates, based on concretization of the relations of belonging and consolidation,[2] at which certain antinomies of set theory and of the broadened calculus are known not to be obtained. This concretization consists essentially of the fact that each of these relations is split up by Bochvar into two, having particular properties (defined by particular axioms). The essence of this concretization consists of the fact that each of these relations is split by D. A. Bochvar into two, which have particular properties (determined by special axioms).

In Section 1 of the article [6] D. A. Bochvar dwells especially on the general method he proposes for the concretization of such calculations, which in models do not define fully uniquely certain of the symbols or combinations of symbols that enter into the formulas of these calculations,) Roughly speaking, the idea which D. A. Bochvar uses to guide himself in this case consists of the fact that not every (single-place, for the sake of simplicity) predicate has a fixed (rigid) volume. The concept of volume: the object x, contained in the volume of the predicate φ, has the property φ; but if the object has the property φ, this still does not signify in general that it is an element of the volume of the

1. Two (single-placed) predicates p and q are called equivalent if one can prove in K_0 the formula
2. The relation of consolidation is a certain generalization of the relation between the predicate and its argument.

predicate φ . We shall interpret the expression «$\varphi(x)$» as the "x has the property φ ", while the expression «$x \in \varphi$» will be treated as "x is an element of the volume φ "[1] and we shall require in particular -- with the aid of the concretization of the consolidation relation -- that (axiom III)[2] in the volume of the predicate there could be no such element ψ , for which $\psi \in \psi$ is true. Then it is clear, for example, that we can introduce the definition of "normal predicate " φ by means of the formula

$$N(\varphi) \underset{Def}{=} \overline{\varphi \in \varphi} \tag{2}$$

(N is the symbol for the individual predicate "normal"), without resorting to the Russel antinomy. Actually, let us insert N instead of φ in (2). We obtain

$$N(N) \underset{Def}{=} \overline{N \in N} \tag{3}$$

From axiom (III) it follows, however, that

$$\psi \in \varphi \rightarrow \overline{\psi \in \psi}$$

or, inserting ψ in the place of φ ,

$$\psi \in \psi \rightarrow \overline{\psi \in \psi},$$

i.e.,

$$\overline{\psi \in \psi};$$

In particular, consequently, $\overline{N \in N}$. The right half of formula (3) is thus proved - consequently the left part is also proved, i.e., $N(N)$. Instead of a paradox in the ("split") calculus, D. A. Bochvar obtains thus a theorem, which states that Russel's paradoxical predicate "does not contain itself as an element" and belongs to itself as a property, i.e., thereby it does not contain itself as an element (is in itself normal). Other antinomies of the same type are similarly resolved in this calculus of D. A. Bochvar.

Let us note, finally, that the set is very naturally defined in [6] in terms of the volume of the predicate, or more accurately, as such a predicate ψ , for which

---

1. In [6], D. A. Bochvar uses other symbols.
2. We give here the rougher formulation of this axiom.

$\phi(x) \equiv x \in \psi$. To the contrary, the a priori distinction of sets and classes, as is done for example in the Σ system of Goedel, which we have discussed above, appears to Bochvar [6] to be less natural.

In further works, reported by D. A. Bochvar at the sessions of the Seminar on Mathematical Logic at the Moscow University (13 November, 18 December, and 25 December 1957), he engages in a logical classification of the formulas of the broadened calculus of predicates relative to the applicability of the "convolution" rule to them, i.e., to the possibilities of using them to define individual predicates and aggregates of individual predicates (from the fact that each individual predicate can be joined to the system without contradiction, it still does not follow as yet that in their aggregate they cannot lead to a contradiction).

However, these investigations are still not completed at the present time.

2. Descriptive Theory of Sets[1]

1. At the initial period of the development of the Moscow School of the Theory of Functions, the descriptive set theory was at the center of the scientific interest of N. N. Luzin and many of his students: M. Ya. Suslin, P. S. Uryson, P. S. Aleksandrov, A. N. Kolmogorov, L. V. Keldysh, M. A. Lavrent'yev, P. S. Novikov and others. The result of the works in the field of descriptive theory of sets, of that period appeared, on the one hand, to be a vigorous development of this field itself, and on the other hand an extension of general concepts, that arose in the descriptive theory of sets, to various other fields of mathematics. In an axiomatic description of any particular system of mathematical concepts, one frequently resorts to the concept of B or A sets. The modern development of the theory of algorithms is to a considerable extent the embodiment of the idea of N. N. Luzin concerning the necessity of investigating the descriptive calssification of denumerable sequences. The close relationships between

---

1. This section was written by Z. I. Kozlova and A. A. Lyapunov.

the concepts of the B and A sets with the concept of recursive and recursive-enumerated sets is universally known. In the most recent time the set-theory concepts have begun to penetrate broadly also in other branches of theoretical natural sciences, in particular in cybernetics. At the same time in recent years the activity in the work in the field of descriptive theory of sets has increased. During the period reported, work in the field of descriptive theory of sets was continued by several of the students of P. S. Novikov and L. V. Keldysh.

Working in Stalingrad are Z. I. Kozlova and I. D. Stupina, in Kolomna -- A. V. Gladkiy, in Ulan-Uda -- B. S. Sodnomov, in Ivanovo -- A. D. Taymanov, in Simferopol' -- Ya. L. Kreynin, in Glazovo -- R. Yu. Matskina, in Riga -- E. I. Arin'. In Moscow problems of descriptive theory of sets are treated sporadically by Yu. S. Ochan', Ye. A. Shchegol'kov, S. V. Yablonskiy, and A. A. Lyapunov.

During the period reviewed, work was carried out in the following directions:

1) A study of the structure of A sets and projective sets was completed in the main outlines even in the preceding period, if one disregards the problems that gave rise to the fundamental difficulties. During the time reviewed, a study was contained of the singularities of the construction of flat sets with specified descriptive nature.

2) Various works were carried out on problems which are affiliated with descriptive theory of sets and abstract topology, and also for descriptive and axiomatic theory of sets.

3) Development was carried out of the theory of operations on sets.

Works on the study of special properties of plane sets have become associated to a considerable degree with works on the theory of operations on sets.

In addition, it is necessary to emphasize the very interesting works on the study of recursive sets, which were carried out by V. A. Uspenskiy and B. A. Trakhtenbrot, which in their contents pertain to the theory of algorithms, but which are very close in their idea to the descriptive theory of sets. We have very little time to prepare the review and

we therefore apologize for its incompleteness.

2. In the works of Z. I. Kozlova [2, 5, 6, 8, 11, 14] and I. D. Stupina [1, 2] an investigation was carried out on the special properties of plane sets with a specified descriptive nature. These works are a natural continuation of the work by N. N. Luzin, V. I. Glivenko, and P. S. Novikov on the study of covering and splitting of plane B sets. The first works by Z. I. Kozlova pertain still to the prewar period. A characteristic of this trend was the establishment of results of the following type.

The set belongs to the absolutely first class, if it has on each compact set a point of local compactness. In this case the given set can be represented as a transfinite sum of compact sets, where each component is separable by means of a set of zero class from the sum of all the following sets.

The minimum length of such a sum is called a subclass of a given set.

In the Baire space J_{xy} there are considered plane A sets $\mathscr{E}$, which have that property, that all sets $\mathscr{P}_x\mathscr{E}$ of the absolutely first class of subclass $<\alpha$ where $\alpha < \Omega$ ($\mathscr{P}_x$ denotes the sets of all the points of space J_{xy} with a constant abscissa x). It is proved that any A set $\mathscr{E} \subset J_{xy}$ of the indicated type can be covered by a B set H (i.e., $H \supset \mathscr{E}$) such that all sets $H\mathscr{P}_x$ are also absolutely of the first class of subclass $<\alpha$.

This theory admits to the following generalization: instead of a set of absolutely first class it is possible to consider such sets, which are expanded in a transfinite sum of subsets, having a compact closure, whereby each component is separable from the sum of all subsequent sets of null class.

For plane B sets H, for which each of the sets $\mathscr{P}_x \cdot H$ is a set of absolutely first class of subclass $<\alpha$, where $\alpha < \Omega$, there exist an expansion in the form

$$H = \sum_{\xi<\alpha} H_\xi,$$

where $H_\xi \cdot H_{\xi'} = 0$, for $\xi \neq \xi'$, each of the components H_ξ is a B set, each of the sets $\mathscr{P}_x \cdot H_\xi$ is compact, and $\sum_{\xi<\alpha} \mathscr{P}_x \cdot H_\xi$ is such that each component is separable from

the sum of all subsequent by means of a set of zero class relative to $\mathcal{F}_x$.

Analogous results are obtained also for cases when $\mathfrak{F} \cdot \mathcal{F}_x$ has certain other topological properties, describable in terms of transfinite indices.

3. Further investigations of problems concerning coverings has followed the line of investigating problems relating with the theory of operations on sets. In the work of A. A. Lyapunov [36] the theorem is established concerning the covering of R_α sets: If N is a rigid base of the R^α operation and $\mathfrak{F} = \{E_{n_1 \cdots n_k}\}$ is a table of R_α sets (or of $R_{\alpha\beta}$ sets) such that each point of the set $R_{\{N\}}\{E_{n_1 \cdots n_k}\}$ is a point of a N-uniqueness of the table $\mathfrak{F}$, then there exists a table $\mathfrak{F}' = \{H_{n_1 \cdots n_k}\}$ of BR_α sets (or $BR_{\alpha\delta}$ sets) such that $H_{n_1 \cdots n_k} \supset E_{n_1 \cdots n_k}$ and each point of the set $R_{\{N\}}\{H_{n_1 \cdots n_k}\}$ is a point of N-uniqueness of table $\mathfrak{F}'$.

Z. I. Kozlova has shown that the theorem remains valid if the points of N-uniqueness are replaced by points of N-p uniqueness and by points of N of finite valuedness.

In the papers by Z. I. Kozlova and I. D. Stupina it is shown that the covering theorem holds also for A-, Γ-, A_2- and CA_2- operations on CA_2- sets, for the cases of points of N-p valuedness, N-finite valuedness, N-non-denumerable-valuedness, and for certain other cases.

4. In the descriptive theory of sets several results were obtained of the following character: one considers a projection of a plane set and expresses an opinion concerning the descriptive nature of the set of points of the projection, the inverse images of which have a certain special property. For example, if one projects a plane A set, then the set of points the inverse images of which contain not less than two points, is the A set (N. N. Luzin), the set of points the inverse images of which contain a non-denumerable set of points is also an A set (V. Serpinskiy).

Many problems of similar character were solved by P. S. Novikov, V. Ya. Arsenin, S. Braun, and K. Kunuguya.

Analogous theorems in the theory of operations have been established by A. A. Lyapunov [36, 51].

Z. I. Kozlova and I. D. Stupina obtained several results, in which they studied the degrees of degeneracy of the results of an operation depending on the topologi-

cal characteristic of the set of chains of the given operation, which specified each individual point.

5. In all these investigations the principal apparatus used was the multiple separability. In this connection, the question arises of the necessity of systematization of the principal premises of multiple separability as applied to various set-theoretical systems. Such an investigation was carried out by Z. I. Kozlova.

Superposing on the system of sets requirements of invariance under certain operations and assuming that in this system there take place only relations of multiple separability, it becomes possible to establish that in this system there arise also certain other relations of multiple separability. A certain summary of the results of such a character was obtained by Z. I. Kozlova [1, 3, 8, 9].

6. Let us now proceed to an examination of works devoted to a detailed study of certain structural properties of sets, studied in the descriptive theory of functions.

A. A. Lyapunov [29, 39, 51] has introduced the concepts of a rarefied class of set and descriptive measurability.

The class Σ of subsets of absolute B-set J is called rarefied, if it satisfies the following conditions:

1°. $J \notin \Sigma$.

2°. The subset of the set that belongs to Σ belongs to Σ.

3°. Any set E, belonging to Σ, is contained in the B-set E*, belonging to Σ.

4°. A sum of not more than a denumerable number of sets belonging to Σ belongs to Σ.

5°. Any system of pairwise non-intersecting B-sets, not belonging to Σ, is not more than denumerable.

The set E, represented in the form $E = E_1 + E_2$, where E_1 is a B set and $E_2 \in \Sigma$, is called Σ-measurable.

A set that is Σ-measurable for any rarefied class Σ, is called descriptively measurable. Certain related properties of sets were considered by E. Spielrein and M. Condo.

Any descriptive-measurable set has the Baire property and is absolutely measurable. In fact, in all those

cases when the presence of the measurability of the Baire property is established along the path of descriptive theory of sets, the descriptive measurability is established.

The problem arose of what is the relationship between the property of descriptive measurability to the absolute measurability and the Baire property. A. V. Gladkiy, leaning on the hypothesis of continuum, has constructed an example of a set that has the Baire property and is absolutely measurable, but is not descriptively measurable.

In the work by Ya. L. Kreynin [1] there were found several general conditions, which are sufficient to make, in a certain abstract space, correct the theorem of A. N. Kolmogorov concerning the non-empty class of sets, obtained by δs -operations. In another investigation he investigated certain effective methods of defining set-theoretical concepts. The concept of effective non-denumerability was introduced earlier by P. S. Novikov, while the concept of effective measurability was investigated by A. A. Lyapunov [26]. Ya. L. Kreynin has shown that any B-set, which is effectively distinct from A-set invariably contains a perfect nucleus.

7. Let us proceed now to an examination of works concerning the study of B-functions.

E. I. Arin' [3] has investigated the construction of B-functions $f(x) = \varphi(x, x, \ldots, x)$, where $\varphi(x_1, x_2, \ldots, x_n)$ is a continuous function in each of its arguments.

S. V. Yablonskiy [1] gave a new exposition of the singular properties of the fundamental properties of B-functions, leaning on the theorem of the separability of B-sets. In the same investigation there was constructed a mutually-unique B-mapping of the Baire space on a Hilbert space, which permits establishing quite automatically many set-theoretical properties of sub-sets of Hilbert space.

8. We now proceed to an examination of problems that are related to descriptive set theory and abstract topology. We note first of all that in the very deep researches of L. V. Keldysh, pertaining to the topology of open representations, a considerable role is played by the methods of descriptive theory of sets and in particular by the use of the theorem concerning the multiple separability

for closed sets.

R. Yu. Matskin [1 -- 7] has investigated the structure of non-continuous and also of continuous and mutually-unique samples of closed sets of Hilbert space, which were found to be arbitrary A and B sets.

A. D. Taymanov [2] gave an analytic representation of rigid bases of $\mathfrak{G}$-operations and has clarified the descriptive nature of rigid bases for certain $\mathfrak{G}$-operations, and specifically has shown that the rigid base of A operations is a set of type G_δ, while the rigid base of a Γ-operation is the CA set.

The works of A. D. Taymonov "On Quasi-Component Non-Connective Sets" [1, 3] arose out of the works of P. S. Novikov, who proved that a set of components of an arbitrary A set has a cardinality that is either continual or not more than denumerable.

A. D. Taymanov generalizes the concept of a quasi-component, introduced by F. Hausdorff, gives a definition of a quasi-component of rank α and defines the index of connective components. Let there be given a non-connective set $E \subset R^{(n)}$. The 1-quasi-component E_x^1 of the point x in topological space E is defined as the intersection of all the open-closed sets in E, containing the point x. The component of the point x in E is contained in the quasi-component E_x^1 and coincides with it when and only when E_x^1 is connective.

The quasi-components of rank $\alpha > 1$ are defined inductively. We assume that there has been defined an α-quasi-component E_x^α of the point x in E for all $\alpha < \beta$, then we determine the β-quasi-component E_x^β. Two cases are possible:

1) β is the number of the first kind, if $E_x^{\beta-1}$ is a connective set, then by definition $E_x^\beta = E_x^{\beta-1}$. If $E_x^{\beta-1}$ is a non-connective set, then the one-quasi-component of the point x in $E_x^{\beta-1}$ is called the β-quasi-component of the x in E and is denoted by E_x^β.

2) β is the number of the second kind. We then put $E_x^\beta = \bigcap_{\alpha<\beta} E_x^\alpha$.

The quasi-components E_x^α are closed in E. The α-quasi-components of one rank of two different points either coincide or else do not intersect, so that one can speak of the breakdown of the set E into α-quasi-com-

ponents and this breakdown, generally speaking, is larger than the breakdown into components.

We denote by K_α^E a set of quasi-components of a rank not exceeding α. A. D. Taymanov proves the following premises:

1) If E is an A-set then K_α^E for $\alpha < \Omega$ either has the cardinality of continuum, or is not more than denumerable.

2) From the existence of an effective set having $\aleph_1$-quasi-components, follows the existence of an effective set having $\aleph_1$ points.

These results take place in arbitrary metric space with a denumerable base.

In addition A. D. Taymanov gave methods for the investigation of the descriptive nature of the space of quasi-components of various sets. Later on in the work, when the topology of closed representations is investigated A. D. Taymanov [8] shows that the closed representations of CA-sets, and also CA_n-sets are sets of the same nature.

He also succeeded in investigating the nature of finite and denumerable-multiple open images of A- and CA-sets.

9. During the last decade many works were carried out on the study of operations on sets, leading to R-sets, and their generalizations (the principal works of A. A. Lyapunov on R-sets [17, 19, 24, 27, 28, 36 -- 39, 51] were carried out in 1946 -- 1947, although they were published in detail only in 1953).

In these works the question was raised of investigating regular processes of complication of operations on sets and clarification of the question at what types of complication of operations on sets are retained very structural properties of sets, obtained with the aid of these operations. It was found that instead of the cumbersome apparatus of the R operations it is possible to use the considerably more flexible and at the same time more readily visualized T-operations.

Let there be given a sequence of bases $\{N_n\}$. A T-operation on an arbitrary sequence of sets $\{E_n\}$, corresponding to this sequence of bases, is defined as follows:

$$E_n^1 = E_n, \quad E_n^{\alpha+1} = E_n^\alpha \Phi_{N_n} \{E_{n'}^\alpha\}_{n'}, \quad E_n^\gamma = \prod_{\alpha<\gamma} E_n^\alpha,$$

where γ is a transfinite number of second kind. Then

$$T_{\{N_n\}}\{E_m\} = \prod_{\alpha<\Omega} E_0^\alpha = E_0^\Omega.$$

A - and R^α -operations can be represented in the form of T-operations. For these operations, one defines in a natural manner transfinite indices and for these indices it is possible to establish the principle of comparison of indices, which generalizes the principle of comparison for A-operations of P. S. Novikov.

If all the operations $\{\Phi_{N_i}\}$ retain a descriptive measurability, then the $T_{\{N_i\}}$ -operation has the same property. If a certain projective class is invariant relative to all operations $\Phi_{\{N_i\}}$, then it is invariant relative to the operation $T_{\{N_i\}}$. The base of the $T_{\{N_i\}}$ operation is obtained in the following manner: let us put

$$H_{n_1 \dots n_k} = \delta_{n_1 \dots n_k} N_{n_k},$$

$$S_0 = N_0, \quad S_k = \sum_{n_1, \dots, n_k} H_{n_1 \dots n_k}, \quad S = \prod_{k=0}^{\infty} S_k.$$

Then

$$T_{\{N_i\}} = \Phi_S.$$

A considerable part of the theory of A-set, and also of C-and R-sets, is obtained by using T-operations.

10. The problem of investigating further types of broadenings of set-theoretical operations arose in a natural manner.

A. A. Lyapunov [39] succeeded in constructing a certain class of set-theoretical operations, which no longer are δs -operations and which have the following properties:

1) All these operations are obtained by means of a regular transfinite process of broadening and are numbered with the aid of transfinite numbers of first, second, and third classes.

2) These operations are characterized by the fact that for these there exists a "feedback" between the

reprocessed sets and the base.

The operations function in such a manner: one considers the inclusion or non-inclusion of a given point in a set of the proposed sequence. From a certain sequence of these sets one choses the final base, and then the operation at a given point is performed with the selected base. Thus, it becomes necessary to act on each individual point. One must note that the entire construction is quite effective.

3) If the initial operation retains the descriptive measurability, then also all operations obtained from it by means of the described broadening also retain the descriptive measurability.

4) R-operations coincide with the first Ω -operations of this system.

All this construction can be considered as a new embodiment of the idea of P. S. Novikov concerning the construction of the maximum-broad system of effectiveness of sets.

The weak spot in this construction is that is relationship with projective sets remains unclarified.

11. Next A. A. Lyapunov [44 -- 46] has constructed certain new processes in the broadening of set-theoretical operations. Let there be given sequenses of sets $\{E_n\}$ and two sequences of bases $\{N_n\}$ and $\{M_n\}$. Let us put

$$E^0_n = E_n, \quad R^{2\alpha+1}_n = E^{2\alpha}_n \underset{m}{\Phi}_{N_n}\{E^{2\alpha}_m\},$$

$$E^{2\alpha+2}_n = E^{2\alpha+1}_n + \underset{m}{\Phi}_{M_n}\{E^{2\alpha+1}_m\}$$

and for numbers γ of the second kind $E^{\gamma}_n = \lim\limits_{\alpha<\gamma} E^{\alpha}_n$. Then

$$T^{\Omega}_{\{N_n, M_n} \{E_m\} = \lim\limits_{\alpha<\Omega} E^{\alpha}_1 = E^{\Omega}_1.$$

The T^{Ω} -broadening has many analytic properties, which are allied to the analytic properties of T -broadenings.

12. The next step in the direction towards broadening the set-theoretical operations was made by Z. I. Kozlova, who started out with works of L. V. Kantorovich and L. Ye. Livenson on the study of projective operations. In the most recent times, Z. I. Kozlova has succedded in

separating a new class of set-theoretical operations, defined in the following manner:

[illegible]

For these operations it becomes possible to define a class of internal transfinite indices, which is fully analogous to the class of minimal indices of P. S. Novikov. For these indices it becomes possible also to obtain a certain form of the principle of comparison of indices, which generalizes the principle of index comparison of P. S. Novikov for A_2 sets. The next problem is a study of the mutual relationships between various types of broadening of set-theoretical operations.

13. During the period reviewed, there was published in <u>Usphekhi matematicheskikh nauk</u> a cycle of articles on descriptive theory of sets, where a summary exposition of the theory of B-sets was given (Ye. A. Shchegol'kov [2]), of A-sets (A. A. Lyapunov [32] and V. Ya. Arsenin [4]), B-functions (A. A. Lyapunov [33]) and the theory of operations on sets (Yu. S. Ochan [13]). In these articles a considerable simplification was obtained of its exposition, and furthermore, it contains many new results. We note the results of Ye. A. Shchegol'kov on that a plane B-set, intersected by any perpendicular to the OX axis along sets of type F_6, is unified by means of a B-set.

In the article by Yu. S. Ochan are expounded original results which concern the comparison of cardinalities of operations with respect to classes of sets, which are invariant relative to a certain operation, for example, the operation of the sieve and the A-operation are equivalent to each other with respect to classes of sets that are invariant relative to finite intersections, but are not equivalent to each other in the general case.

14. Concluding the survey of work on descriptive theory of sets, it is necessary to note with satisfaction, that during the reviewed period there were published the lectures by N. N. Luzin "On Analytic Sets" in two volumes of his collected works, in which are included papers on descriptive, metric theory of functions, and also papers on the theory of functions of complex variable.

Chapter II

THEORY OF ALGORITHMS AND COMPUTABLE FUNCTIONS AND OPERATORS

3. Representation of Recursive Functions Functions of Large Spread

1. The creation of a meaningful mathematical theory of constructive objects of mathematics began naturally with a study of computable functions (of a finite number of variables), as arguments, since the values of which are the natural numbers 0, 1, 2, 3,...[1] Since the very construction of the natural numbers has a recursive[2] character (in order to determine, for example, the number five, it is necessary to define first the number 4; to define the number 4, it is first necessary to define 3, etc., down to 0, which is considered as directly defined), then it is natural to identify the computable functions with recursive functions (Church's thesis).

Even the simplest recursive (inductive) definitions (of a single-place) function f have an implicit character; the value of the function for any non-vanishing value of the argument is defined in terms of the value of the same function for the preceding value of the argument. The most general definition of a recursive function is obtained as a generalization of the following: the function is called recursive if it can be specified by a system of equations which not more than uniquely determines its values and which

---

1. See the foreword by A. N. Kolmogorov to the book by R. Peter "Recursive Functions," Moscow, Foreign Literature Press, 1954. From here on we shall call functions of this type arithmetic or numerical.
2. In the literal sense of the word (recurso -- latin for running back, return) a recursive definition is naturally called a definition which is realized with the aid of "returning" from the unknown to the known (A. N. Kolmogorov, ibid, p. 4).

permits their calculation (for these values of the arguments, for which the function has been defined) in terms of the values of the function itself (and of other functions, which can be determined from it simultaneously by the same system of equations), i.e., it thereby bears a more implicit character. (It is naturally understood that if for certain given values of all its arguments the value of the function is determined by a corresponding system of equations in terms of its value for the same values of the arguments, then for these values of the argument the function is not defined.)

A recursive function that is defined everywhere is called generally-recursive. If a recursive (n-place) function is not required that it be defined for all the groups of n values of its arguments, then the function is called partially-recursive. In the class of generally-recursive functions one separates usually, as the simplest subclass, the class of primitively-recursive functions, which contain the constant 0, the succession function (which relates to the number n the succeeding number n + 1) and closed relative to the operation of substitutions[1] and schemes of primitive recurrence (which define the value of the function f for the arguments $(n+1, x_1, \ldots, x_n)$ as the value of the already defined function φ for the arguments $(n, f(n, x_1, \ldots, x_n), x_1, \ldots, x_n)$ and which specify directly the value of the function f for the arguments $(0, x_1, \ldots, x_n)$ as the value of the already defined function ψ for the same arguments).

Through Goedel's arithmetization of logical and logical-mathematical calculations (generally, so-called formalized systems), primitive-recursive functions acquire a particular significance in the construction of the gene-

1. By "substitution" we mean here both the substitution of functions, as well as the substitution of variables. If one does not use the operation of substitution of variables, then it is necessary to add to the initial functions in this definition also the function of identity $I(x) = x$. V. A. Uspenskiy's refinement of the general concept of substitution can be found in the book by R. Peter, "Recursive Functions, Moscow, Foreign Literature Press, 1954, p. 38 (footnote).

ral theory of such systems. Even the very method of specifying primitive-recursive functions recalls the method of specifying the rules for the production and rules of transformation in logical calculations: one specifies directly the primitive-recursive functions as initial ones, and inductive rules are then given which makes it possible from the already-constructed primitive-recursive functions to construct new ones. However, the imparting of an exact meaning to this analogy and the clarification of the actual meaning of primitive-recursive functions is a difficult problem certain results of which will be clarified by us in connection with the work of A. V. Kuznetsov [1], based on a study of the examples of complex-recursive functions (i.e., general-recursive functions which are not primitive-recursive), belonging to the author.

Let us note that in the history of the development of the theory of recursive functions and operators, the principal significance was attached initially only to general-recursive functions: the most important role of partially-recursive functions (and particularly operators) was clarified completely with active participation of the Soviet mathematicians and logicians, which we shall dwell on later, only in recent years.

The implicit character of the specification of the recursive function by means of a system of equations that defines it has naturally given rise to the desire for replacing this implicit definition by an explicit one. The latter was realized for generally-recursive functions in 1936, for partially-recursive functions in 1943 by Kleene, who has shown that is enough to add to primitive-recursive functions (and predicates) the operator «μy...» "the smallest y such that..." in order to obtain the possibility of representing explicity any partially-recursive function, specified (implicitly) by a system of equations that defines it. Connected with the Kleene representation is one important problem a complete solution of which, now contained in all the monographs and handbooks on the theory of recursive functions, belongs to A. A. Markov. We now proceed to an elucidation of the corresponding works by A. A.

Markov of 1947 [36] and 1949 [39][1].

2. The explicit representation proposed by Kleene for a recursive function F of n arguments has the form

$$F(x_1, \ldots, x_n) = P(\mu y(Q(x_1, \ldots, x_n, y) = 0)), \tag{4}$$

where P is the primitive-recursive function of one argument, Q a primitive-recursive function of n + 1 arguments, $\mu y \ldots$ -- the smallest of the numbers y which cause $Q(x_1, \ldots, x_n, y)$ to vanish (for given $x_1, \ldots, x_n$). (If F is a general-recursive function, then the function Q should satisfy the additional condition

$$\forall x_1 \ldots \forall x_n \exists y (Q(x_1, \ldots, x_n, y) = 0),$$

i.e., the function $\mu y \ldots$ should be defined for all the groups of n of $(x_1, \ldots, x_n)$.)

It was found here (this was clarified by Kleene in 1943) that in representation (4) it is possible to choose the primitive-recursive function P quite independent of the represented function F. However, the function P, which is universal in this sense, and indicated by Kleene himself, was defined by means of a very complex aggregate of substitution schemes and primitive recursion schemes.

In 1944 Skolem suggested that one can dispense in general with the function P, i.e., one can represent any general-recursive function F of an argument in the form

$$F(x_1, \ldots, x_n) = \mu y(Q(x_1, \ldots, x_n, y) = 0), \tag{5}$$

where Q is a primitive-recursive function of n + 1 arguments, satisfying the foregoing additional condition.

Skolem formulated in this case a simple necessary and sufficient condition for representability of a general-recursive function in the from (5). However, the prob-

---

1. The paper by A. A. Markov [39] contains a detailed proof and a certain strengthening of the result detailed in his note [36]. The solution of Markov together with a complete proof of all his theorems is given in the book by R. Peter "Recursive Functions", Moscow, Foreign Literature Press, 1954, pp. 190 -- 195. See also S. C. Kleene, Introduction to Meta-Mathematics, Moscow, Foreign Literature Press, 1957, p. 258.

lem of whether any general-recursive function satisfies this condition has remained open.

In 1946 Post answered this question in the negative, giving an example of a general-recursive function that did not satisfy the Skolem condition.

A complete solution of the problem of the class of primitive-recursive functions P, which can play a role of universal ones in the representation (four) was given in 1947 by A. A. Markov [36].

Markov's solution consists of the following.

A primitive-recursive function P (of one argument) is universal when and only when it is a function of large spread, i.e., it assumes all the natural values, and with this each of this it assumes an infinite number of times. In the representation of Skolem (5) when actually he takes for P the function $P(x) = x$, which is not a function of large spread (it assumes each value only once). By virtue of the Markov theorem this explains that not every general-recursive function is representable in the form (5).

We remark that the proof of necessity for a universal (for all general-recursive) functions P be a function of large spread[1] is of the reverse type, i.e., not constructive. "It is hardly possible to replace it by a constructed proof," notes A. A. Markov [39] (p. 424), and he formulates on the spot a weakened form of theorem III (equivalent to its double negation), proved constructively by his arguments.

3. In the same work A. A. Markov [39], in connection with the proof of the lemma he requires raised a question equivalent, as noted by A. V. Kuznetsov [1], to the following: do there exist such monotonically-increasing complex-recursive functions $\varphi(x)$, for which the set of values are primitive-recursive (i.e., primitive-recursive predicates F, where $F(x) \equiv \exists t\,(\varphi(t) = x)$?

1. We have in mind here the proof of theorem III: on the possibility of constructing such a general-recursive function of n arguments, that in any of its representation in form (4) (with primitive-recursive P and Q) P is a function of large spread.

As is well known, the hanging of an existence quantor on a two-place primitive-recursive predicate gives rise to a one-place predicate, which occupies in the well-known Kleene-Mostowski classification a higher place: this is a recursively-enumerable, but in the general case not a recursive predicate.[1] It turns out, however, as was shown by A. V. Kuznetsov [1][2]), that a positive answer should be given to the foregoing question.

In the fall of 1947 A. V. Kuznetsov independently of W. Ackermann (the example[3] constructed by the latter was unknown at that time to A. V. Kuznetsov) constructed several examples of such functions, which are generally-recursive, but increase more rapidly than all the primitive-recursive functions. One of the simplest examples of such types of function was the function $\varphi(x)$, defined by the following system of equations

$$\varphi(x)=\Phi(x,\ x), \qquad \Phi(x,\ 0)=2x,$$
$$\Phi(0,\ y')=1, \qquad \Phi(x',\ y')=\Phi(\Phi(x,\ y'),\ y).$$

The function $\Phi(x,\ y)$ was later on denoted by A. V. Kuznetsov by 2^x_y ; with this, the result was $\varphi(x)=2^x_x$.

As can be seen from the system of equations defining it, the function $\varphi(x)$ is general-recursive. At the same time, 2^x_x increases with increasing x faster than any primitive-recursive function: no matter what the primitive-recursive function f(x) , we have for it

$$\exists t \forall x\,(x > t \rightarrow 2^x_x > f(x))$$

Thus, 2^x_x is a general-recursvie but not a primitive-

---

1. Recursive predicates are analogues of B-sets, while recursive enumerable ones are analogues of A-sets.
2. The note by A. V. Kuznetsov [1] was dictated on 12 January 1958. The actual results published in it were obtained in 1947--1949.
3. W. Ackermann, Zum Hilbertschen Aufbau der reelen Zahlen, Math. Ann. 99 (1928), 118 -- 133. The works of R. Noether and R. M. Robinson, mentioned in the book by R. Peter "Recrusive Functions," Moscow, Foreign Literature Press, 1954, p. 92, footnote 2, were also unknown by A. V. Kuznetsov.

recursive function.

In general, from a comparison of the function 2^x_n with the primitive-recursive functions, A. V. Kuznetsov obtained in 1947 -- 1948 the following propositions.

a) The function 2^x_n for any (fixed) n is primitive-recursive.

b) For any primitive recursive function f(x) there exists such natural numbers n and m, that

$$\forall x (f(x) < 2^x_n + m).$$

c) In order for the function f(x) to grow more rapidly than all the primitive-recursive functions, it is necessary and sufficient that it grow more rapidly than all functions of the type 2^x_n (n fixed).

With the aid of function 2^x_x , A. V. Kuznetsov indeed answered directly the problem raised in 1949 by A. A. Markov. Namely, it was found that the function 2^x_x (which is monotonically increasing) is generally-recursive, but not primitively-recursive, while the predicate F, where $F(x) \equiv \exists t (2^t_t = x)$ is primitive-recursive (A. V. Kuznetsov [1]).

From among the number of other applications of everywhere computable (general-recursive) functions, which increase more rapidly than all primitive-recursive ones, with which A. V. Kuznetsov dealt in 1947 -- 1948, we shall mention the following.

The first chapter in the booklet by A. Ya. Khinchin "Tri zhemchuzhiny teorii chisel" [Three Pearls from the Number Theory] is devoted to a proof of the well-known Van der Waerden theorem, which states:

Let k and l be arbitrary natural numbers. Then there exists such a natural number $n(k,l)$, that when any segment of a series of natural numbers of length $n(k,l)$ is broken up in any manner into k classes (among which there may be empty ones) an arithmetic progression of length will be found in at least one of these classes.

In two different editions of the booklet by A. Ya. Khinchin, this theorem is proved differently. But in both cases the function $n(k,l)$, which satisfies the conditions of the theorem, is so constructed, that it isn't majored by any primitive-recursive function. The latter was

indeed observed by A. V. Kuznetsov with the aid of estimates, obtained by him through comparison of a defining construction for $n(k, l)$ with the definition of 2^x_y .

In addition to the answer to the question of A. A. Markov, the note by A. V. Kuznetsov [1] contains a certain attempt to develop a general theory of functions of large spread and the associated (with these functions and not with the expansions into simple factors) numerations of pairs, groups of n, and processions (i.e., any type of finite sequences).

In the note by A. V. Kuznetsov [1] there is alo formulated a certain thesis, pertaining to the characteristic features of any effective computation process, and as a consequence of this a general conclusion is drawn concerning an estimate of the length of the process of calculation of a complex-recursive function.

As is known, the question of any particular characteristic differences of primitive-recursive functions from general-recursive functions has not found as yet a convincing solution. It is true, naturally, that the remark of R. Peter[1] (made by her in connection with the foregoing Ackermann example), that the most substantial peculiarities of recursive functions, which are not primitive-recursive, appear indeed in examples of functions which are not majored by any primitive-recursive functions, is correct. But the question of whether it is necessary to separate as simplest functions all the primitive recursive functions still remains unclarified to date. The thesis of A. V. Kuznetsov casts light on the role of primitive-recursive functions in any algorithmic (realized in accordance with definite rules) process of calculation. It pertains to elementary steps, into which this process of calculation is broken up but which themselves no longer lend themselves to further subdivision, and consist of the following: the rules of the algorithms can always be formulated in such a way, that each elementary step consists of a (single) application of such rules of computation (analogues of the rules of deduction), of which there is a finite number and each of which is such, that the relation between that which is

1. "Recursive Functions," Moscow, Foreign Literature Press, 1954, p. 182.

obtained from a given rule with that from which we obtain it (upon single application of the rule), is (after corresponding arithmetization) a primitive-recursive predicate. The consequence of this thesis[1] is such: The length of the process of calculation (expressed, for example, by a number of symbols, which must be written out during the calculations, or by a length of time) of a complex-recursive function in accordance with no matter what type of algorithms is not measured by any primitive-recursive function. The unclarified question, which was mentioned above (and which was formulated by A. V. Kuznetsov in January 1949) pertains to whether it is possible, in the formulation of the foregoing thesis, to limit still further the class of predicates, corresponding to the elementary (non-divisable) steps of the calculation. Its solution was prevented by the absence of a single visualization of all the conceivable elementary steps of an arbitrary algorithm. At the present time such a visualization can be obtained by starting with the definition of the algorithm after Kolmogorov, which will be discussed in the next section.

4. Definition of the Algorithm. General Theory of Algorithms

1. Church's thesis identifies calculable functions with the recursive functions. However, the concept of a recursive function arose not as an indirect reflection of the process of calculation itself of the value of the function, starting with a system of values of its argument. From any definition of any kind of recursive function (implicit, and more so, explicit) it is easy to extract, naturally a method of constructing the algorith, which converts each specified system of values of the arguments (for which the function is defined) into corresponding values of the function. But the definition of the recursive function is in itself not yet an algorith. Furthermore, not every algo-

1. Reported at the Seminar on Mathematical Logic at the Moscow University in January 1949.

rithm (a process performed in accordance with an exact prescription and leading from initial data, which may vary, to the sought result (A. A. Markov)) pertains directly to the calculation of the values of arithmetic functions.

With the aid of the methods of arithmetization, the most important of which is the Goedel arithmetization of meta-theories, based on an effective recount of the formal objects of theory, it is possible, naturally, to reduce the process of the existence (or respectively non-existence) of the sought algorithm to the problem of recursiveness (respectively, non-recursiveness) of a certain airthmetic function. It was exactly in this way that the famous proof was obtained (Church, 1936) of non-existence of an algorithm, which permits to recognize, from the form of the formula of a narrow calculus of predicates, whether this formula can be proved in this calculus or not (the problem of resolvability for a narrow calculus of predicates). The same method was used by A. A. Markov, Post, and others to obtain a series of results (which will be discussed in detail in Section 9), pertaining to the impossibility of certain algorithms[1] in theories of associative systems with integral-number matrices.

It is possible to say therefore that the definition of the recursive function makes it possible to assign an exact meaning if not to the concept of algorithm itself, then to statements concerning the existence and particularly the non-existence of an algorithm. From among the other refinements of the concept of algorithm (through the Turing machine, the finite combinatorial process of Post, the definition of a computable function with the aid of the Church λ-conversion calculus, or, more generally, the definition as a function whose values can be derived in a certain logical calculus,[2]

1. The Russian words "algorifm" and "algoritm" are used as synonyms. A. A. Markov and his students usually write "algorifm." In papers by other authors one encounters more frequently the spelling "algoritm."
2. Including also the definition of the algorithm in terms of simulation by finite classes, proposed in 1949 by B. A. Trakhtenbrot [1, 6] which we shall deal with in Section 12.

etc.), the closest to a description of any automatically performed process of calculation is the definition of the Turing machine; the others either do not satisfy (analogously to the definition of the recursive function) directly the question "what is an algorith?" or else describe directly only certain types of algorithmic processes, which are realizable, for example, by means of a machine of definite construction. However, one can judge that even in the Turing machine there are not exhibited directly the characteristic features of any effective automatic process, performed in accordance with a definite program of action on initial data that are capable of varying, at least from the fact that the proof of the equivalence of other definitions of the algorithm to the definition with the aid of the Turing machine requires sometimes great cleverness. In this connection, the need arises naturally for giving such a definition (sufficiently accurate, so it be useable in mathematics) of an algorithm, which would be free of all these shortcomings.

In 1951 A. A. Markov [41] proposed a refinement of the concept of algorithm ("normal algorithms" of A. A. Markov), based on the representation of constructive objects of mathematics in the form of words in a certain finite alphabe,[1.] and programs of action of the algorithm in the form of a list of prescriptions written in a definite sequence, requiring the replacement in an already available word of the first entrance of some word P on to a word Q.

The exact definition of normal algorithm proposed by A. A. Markov is constructed, on the basis of the following three characteristic features of any algorithm ([41], p. 176):

a) The presence of an exact prescription, which leaves no place for arbitrariness in the known generally accepted sense -- determiniteness of the algorithm.

b) Possibility of starting with initial data that can vary within certain limits -- mass nature of the algo-

1. If the alphabet contains ten Arabic numbers, then the writing of the number in decimal system is also a word in this alphabet.

rithm.

c) The trend of the algorithm towards obtaining a certain sought result, the final analysis obtained under suitable initial data - resultativeness of the algorithm.

Another principal these, from which A. A. Markov starts, consists of the fact that "any computing process, used in mathematics, reduced to a certain potentially realizable process of successive transformation of words in a suitable alphabet" ([41], p. 180).

Even closer to any real process of calculation, realized by any sum of computing mechanism, is the definition of the algorithm proposed in the winter of 1951 -- 1952 by A. N. Kolmogorov [135]. A student of Kolmogorov, V. A. Uspenskiy, refined for the first time (1955) a concept of a program (algorithm) independent of the choice of the definition of the algorithm, as will be discussed in Section 5. The best developed at the present time is the theory of normal algorithms of Markov. We shall start with this theory the evaluation of the work of Soviet mathematicians, devoted to a refinement of the concept of the algorithm and the general theory of algorithms.

2. Markov's theory of algorithms has been expounded in all its details in a large book [48]. A clear and accessible brief exposition of this theory is contained in the article [41, 46]. The best that we could do here in order to explain the contents of this theory, would be to restate these articles by A. A. Markov. We therefore think it better to refer the reader to them directly.[1] We shall dwell only on the follwing instances here.

a)Inasmuch as A. A. Markov starts with the fact that any algorithmic process used in mathematics consists of converting words in a certain alphabet into words,[2]. he dwells

---

1. A good explanation of the contents of the book of A. A. Markov [48] can be found by the reader also in the review of N. A. Shanin (Referat Zhur Matematika, 1956, Abstract No. 2716.

2. If the prescriptions of the algorithm are such that they cause an unlimited continuation of the reprocessing of the word P, then the algorithm is considered as unapplicable (cont.)

in detail on a general description of the symbol of the alphabet ("letter") and the "abstract letter" and to a "word" compiled of such symbols as the constructive objects, in the construction of which an important role is played only by the proposition that it is possible to distinguish strictly and to identify the initial objects ("abstraction of identification") and to create from them words of any desired length ("abstraction of potential realizability").

The abstraction of actual infinity is not admitted. For what is coming, it is important to distinguish between the terms "algorithm <u>in</u> the alphabet A" (under which is meant "a generally understood prescription, which determines the potentially realizable process on abstract words in A, starting with any word A" [41], p. 180), and "an algorithm <u>on</u> the alphabet A", i.e., an algorithm in a certain alphabet containing A.

b) The normal algorithms of A. A. Markov are specified by schemes of substitutions, i.e., by a list of elementary operations of local character arranged in a definite sequence and performed on words (the "locality" of the operation consists of the fact that the latter touches only on a given type of a section of the converted word).

c) The question as to what extent the exact concept of a normal algorithm corresponds to the previously-formulated general concept and to the not quite exact concept of the algorithm in a given alphabet" ([fl], p. 183, and here, Section 1) the answer is given in the form of the following principle of normalization of algorithms: any algorithm in the alphabet A is fully equivalent with respect to [1] A to a certain normal algorith on A [48]. In confirming the correctness of this statement many proofs are brought, of which the most important consists of the fact

(Footnote 2. on pg. 48 cont.) ...to the word P. In this connection one can say that the algorithm is a partial function, pertaining to the words of the word.

1. Two algorithms are fully equivalent relative to the alphabet A, if every time that one of these processes a certain word P into A into a word Q, the other does the same.

that "all the algorithms thus far known in mathematics are equivalent to normal algorithms" ([41], p. 183). Also in favor of this statement are many theorems on different combinations of normal algorithms. The algorithm obtained as a result of such combinations always is found to be equivalent to a certain normal algorithm.

d) Inasmuch as the scheme of prescriptions that specifies a certain normal algorithm in alphabet A can itself be written in the form of a word in a certain alphabet, then it becomes possible to construct a universal normal algorithm, the application of which to a word representing a pair of words: 1) writing ("image") of a normal algorithm $\mathfrak{A}$ in the alphabet A, and 2) a word P in the same alphabet gives the same as the application of the algorithm $\mathfrak{A}$ to the word P. This theorem on the universal algorithm serves as a base for many proofs of impossibility of algorithms. With its aid it is proved above all (by means of an argument similar to the "diagonal method" of G. Cantor), that it is impossible to have a normal algorithm on an alphabet A_0 (consisting of a pair of letters (a, b)), recognizing the non-self-applicability of the algorithm (i.e., applicable to those and only to those recordings of normal algorithms in A, which are recordings of algorithms which are not applicable to their own recording). Theorems of this type indeed serve as a starting point for the second part of the book [48], devoted to the proof (by now using the means of the general theory of normal algorithms) of the non-solvability of several algorithmic problems of algebra,[1] above all the theory of associative calculus.

(The associative types of calculus are related with the normal algorithms by means of the following theorem: "No matter what a normal algorithm $\mathfrak{A}$ in the alphabet A, it is possible to construct such anassociative calculus $\mathfrak{B}$ on the alphabet $\mathfrak{A} \cup \{\alpha,\beta,\gamma\}$ (here α,β,γ are letters that do not enter into A), that the equality[2] $\mathfrak{A}(P)=Q$ will take place for the words P and Q in the alphabet A when and only when $\beta\alpha P\beta$ is equivalent to $\beta\gamma Q\beta$ in the

1. That is, the proof of non-existence of certain algorithms.

2. $\mathfrak{A}$ (P) denotes of the result of the application of the algorithm $\mathfrak{A}$ to the word P (if such exists).

calculus $\mathfrak{B}$ " [48], p. 208.) Translated into the language of the theory of recursive functions, this means, approximately, that the set of proved equivalences of associative calculus is recursively enumerable, but not necessarity recursive by all means, and therefore the problem of solvability in general case is not solved for it.)

3. The proof of the equivalence of the principle of normalization of A. A. Markov and Church's thesis on the coincidence of the concepts of effective calculability in the general recursiveness was first obtained by V. K. Detlovs [1]. V. K. Detlovs writes the natural numbers in alphabet $\{1\}$ and the ordered groups of n of the natural numbers in the alphabet $C = \{1, *\}$. The word $x_1 * \ldots * x_n$ in this notation an ordered n group $x_1, \ldots, x_n$.

The function $\varphi(x_1, \ldots, x_n)$ is called algorithmic if there exists a normal algorithm $\mathfrak{A}$ on the alphabet C such that

$$\varphi(x_1, \ldots, x_n) \simeq \mathfrak{A}(x_1 * \ldots * x_n).$$

The symbol $\simeq$ denotes that if one of the parts of the formula makes sense, then the other one also makes sense, and both are then equal to each other. If both parts of the formula make sense (and are equal) for all groups of n of the natural numbers, then the function is called fully algorithmic.

The principal result of V. K. Detlovs consists of that the contents of the concepts "algorithmic" and "partially recursive" (functions) coincide. Equally identical are the volumes of the concepts "fully algorithmic" and "generally recursive" functions.[1]

Other participants in the Seminar on Mathematical Logic at the Leningrad Division of the Mathematical Institute imeni V. A. Steklov, namely, N. M. Nagornyy, E. S. Orlovskiy, G. S. Tseytin (students of A. A. Markov),

---

1. The paper of V. K. Detlovs was published, together with complete proofs, "Equivalence of Normal Algorithms and Recursive Functions" and those mentioned later, in the collection of paper of the Leningrad Seminar on Mathematical Logics(Trudy MIAN im. Steklova [Works of the Mathematics Institute of the Academy of Sciences imeni Steklov] 52).

obtained many results pertaining to a further development of the theory of normal algorithms.

In the note [1], N. M. Nagornyy (1953) strengthend the theorem proved by Markov concerning the reduction of normal algorithms, according to which any normal algorithms on a certain alphabet can be replaced by an equivalent (with respect to A) normal algorithm in a merely two-letter expansion of the alphabet A. N. M. Nagornyy has shown that it is possible to confine oneself also to a one-letter expansion, but to go without any expansion of the alphabet at all it is impossible for normal algorithms in the general case. (This is possible for Turing machines, where the empty place between words can be used as a separate symbol[1].) Thus, in particular, a doubling normal algorithm on A is not equivalent relative to A to any normal algorithm in A. This, as proved later on by N. M. Nagornyy [3] is found to be generally true for any normal algorithm which stretches words (by a factor of several times).

In reference [2] (which is expounded in detail in [4]) N. M. Nagornyy considered certain generalizations of the concept of normal algorithms. The principal feature of the algorithms considered by him consist of that each step of the work of the algorithm is not only developed a certain word, but a scheme is indicated, which must be applied in the next step. For the simplest of these generalizations -- for algorithms of type σ, as they are called by the author, it is particularly easy to construct concrete algorithms and the author proves theorems on the composition of algorithms: formation of complex algorithms from those already available. The equivalents proved by the author of the concepts of algorithms of the types considered by him to the concept of the normal algorithm (which is in itself of interest as a supplementary argument in favor of the principle of normalization) makes it possible to use algorithms of type σ for simplification of the proof of many theorems on

1. This result was reported at the Session of the Seminar on Mathematical Logic at the Moscow University in April 1957. It is published in the journal Zeitschrift Math. Logik u. Grundlagen Math.

the composition of normal algorithms.

In a diploma thesis, V. S. Chernyavskiy[1] considered a subclass in the class of normal algorithms, consisting of algorithms which he called a "shuttle" algorithm, and for which he proved that any normal algorithm <u>in</u> the alphabet A is equivalent relative to A to a certain shuttle algorithm <u>on</u> A. By virtue of this equivalence the theory of normal algorithms can be reduced to the theory of shuttle algorithms, in which a series of theorems (particularly on the composition of algorithms, on the equivalence (in a definite sense) of the definition of the shuttle algorithm to the definition of the Turing machine, and on the universal algorithm) are proved more simply (and more uniformly) than in the general theory of normal algorithms.

In the list of prescriptions, which specify the normal Markov algorithm, individual prescriptions (steps of the algorithm) can be of quite different difficulty: for their performance it may be necessary to have as long a time interval as convenient (this will take place, for example, if we deal with ever lengthening words, in which we must search consecutively the first entrance of any definite words or to verify the non-existence of such entrances).

A table of prescriptions defining a shuttle algorithm is used at each step to indicate directly a section of the word, subject to transformation, and this section consists always of not more than two letters. The individual steps in shuttle algorithms are thus (in principle) all of the same difficulty.

Shuttle algorithms are readily interpreted in the form of a process that is realized by a machine of special type or by a living being, i.e., satisfy the requirement that is imposed on a good definition of an algorithm, that this definition directly reflect the course of the performance of the algorithmic process.

4. An estimate of the number of steps in the use of a normal algorithm of Markov, as applied to any word in a given alphabet, was the subject of a paper by G. S. Tseytin,

---

1. In press.

reported by him at two Sessions of the Seminar on Mathematical Logic at the Moscow University (14 November and 21 November 1956).

Introducing natural (inductive) definition of the function $\mathfrak{N}(\mathfrak{A}, P)$ -- the number of steps in the application of the algorithm $\mathfrak{A}$ to the word P (up to termination of the process of conversion of this word), G. S. Tseytin shows above all that for all: alphabets A and (always applicable) algorithm $\mathfrak{A}$ on A, and for any n there exists such an algorithm $\mathfrak{B}$ on A, equivalent to the relative algorithm $\mathfrak{B}$ on A, that for any work P in the alphabet A

$$\mathfrak{N}(\mathfrak{B}, P) < \frac{1}{n}(\mathfrak{N}(\mathfrak{A}, P) + [P]) + 2$$

(Here $[P]$ is the length of the word P).

Inasmuch as the problem of the estimate of the number of steps is posed not for the algorith, but for the algorithmic function (obtained as a result of identification of the equivalent algorithms), where by the estimate is carried out for those (of the equivalent) algorithms, the application of which requires the smallest possible number of steps, it is clear from the foregoing theorem that the only thing of interest is an estimate of the order of the rate of growth ot the number of steps of the algorithm as the length of the word processed by it is increased.

Since in any finite alphabet A there exists only a finite number of words, the length of which does not exceed a given number n; then for each algorithm $\mathfrak{A}$ in the alphabet A it is easy to construct an algorithm $N_{\mathfrak{A}}$ such that

$$N_{\mathfrak{A}}(n) = \max_{[P] \leq n} \mathfrak{N}(\mathfrak{A}, P).$$

It is thus found that if $\frac{n}{N_{\mathfrak{A}}(n)}$ grows without limit with increasing n, then $N_{\mathfrak{A}}(n)$ grows not slower than a certain linear function of n (i.e., it cannot grow, for example, as log n or any similar function, which grows slower than any linear function).

We shall employ the term "regulator" to non-diminishing functions of integers with integral values (more accurately, normal algorithms defining such functions). Then one can naturally say that the number of steps of an

algorithmic function is majored by regulator f, if for any algorithm $\mathfrak{A}$, corresponding to this function, the number of steps is majored by f (i.e., $N_{\mathfrak{A}}(n) \leqslant f(n)$, starting with a certain n). In order to exclude the possibility of a trivial estimate of the number of steps of an algorithmic function, based on the length of the word, which is the value of this function, we shall consider such algorithmic functions, which can assume only two different values (algorithmic predicates). We shall say that the regulator f_1 is "stronger" than the regulator f, if there exists an algorithmic predicate, the number of steps of which is majored by f_1, but is not measured by f. Then the last result shows that if f is a constant and f_1 is "stronger" than f, then f_1 increases not slower than a linear function, i.e., we have here a "jump" from a constant to a linear function. For regulators, on the other hand, which increase no slower than the linear function, the situation is quite different, as shown by the following theorem.

Theorem. Let there be given regulators f and φ, with $f(n) \geqslant n$ and $\varphi(n) \to \infty$. Then it is possible to construct an algorithmic predicate, the number of steps of which is not majored by f, but is majored by the regulator

$$f(n) \cdot [\log_2 f(n)]\, \varphi(n) + \mathfrak{R}(f, n) + \mathfrak{R}(\varphi, n).$$

Dispensing with second-order terms, we can indicate thus that if f increases not slower than the linear functions, and φ increases without limit (although no matter how slowly), the regulator $f(n) \cdot [\log_2 f(n)] \cdot \varphi(n)$ is "stronger" than the regulator f.

5. As already noted, at the present time there are known many definitions of the algorithms, which have a sufficient degree of clarity and generality for mathematical purposes, but at the same time are neither direct refinements of the intuitive concept of the algorithm, or else describe only a certain definite class of algorithms: the thesis connected with each of these definitions and which states that each mathematical algorithm is equivalent to a certain definite class of algorithms, is based most convincingly on the fact that all these different definitions of the algorithm are found to be equivalent.

Could one however give a definition of the algorithm which would refine directly the very idea of the algorithmic process in general (and not only a certain particular type of these processes)?

Such a definition of the algorithm was proposed by A. N. Kolmogorov [135]. This definition is based on the idea of the algorithmic computability, which differs from the computability by means of ordinary (real) computing mechanism only in the unlimited volume of the "memory."

In the algorithm of A. N. Kolmogorov the problem is specified in the form of a one-dimensional topological complex, and the solution is also obtained in the form of a complex, whereas each step of the algorithmic process consists of processing one complex into another in accordance with definite rules of processing, which touch, generally speaking, not the entire complex, but only the surveyable part, the value of which cannot exceed a previously established limit.

Since from the definition of the computable function one can always extract a definition of the algorithm (which realizes the calculation of the values of the function, starting with values of its arguments), so also, to the contrary, a definition of an algorithmic function corresponds to any definition of the algorithm.

By virtue of the generality of the definition of the algorithm in the sense of Kolmogorov, one writes automatically in his terms all other definitions of the algorithm, or more accurately, the algorithms in the sense of other definitions; with its aid it is easier to prove many general properties of algorithms and algorithmic functions in general.

At the same time, the algorithm of A. N. Kolmogorov, as expected, for all its generality, is not broader than the ordinary partially-recursive functions. Actually, to each Π -complex (complex with a surveyable portion segregated in it in a definite manner) one can place in correspondence effectively and mutually uniquely a natural number. The function $L = \Gamma(K)$ (which relates complexes to complexes and generally speaking, which is not everywhere defined) therefore induces in the natural series the function $l = \gamma(k)$ (which also, generally speaking, is not everywhere defined). With this, as shown by V. A.

Uspenskiy,[1] if $\Gamma(K)$ is an algorithmic function, then $\gamma(k)$ is a partially recursive function. The inverse, true, is not directed correct, since not for every partially-recursive function $\gamma(k)$ there exists corresponding algorithmic function $\Gamma(K)$ (thus, in the general case $\Gamma(K)$ does not exist, if $\gamma(k)$ is defined on the entire set of number of complexes). V. A. Uspenskiy[1] has shown, that nevertheless for any function in the natural series $\gamma(k)$ one can still find a representative $\Gamma(K)$ which is furthermore a very natural one) among the functions of the complexes in such a way, that if $\gamma(k)$ is a partially recursive function, then $\Gamma(K)$ is an algorithmic function.

The equivalence of the definition of the algorithms of A. N. Kolmogorov to the definition of the computable function as a partially recursive one, and thereby to other known definitions of the algorithm, was thus proved by V. A. Uspenskiy.

5. Enumerable /Countable/ Sets and Computable Operations on Sets

General Concepts of Enumeration and Programs.

The theory of enumerable or, as they are otherwise called, recursive-enumerable sets is of particularly great significance for mathematic logic and meta-mathematics. Enumerable sets correspond in their content to sets of constructive objects, which are generated by some effective regular procedure. In order to be able to speak of a certain object, it is necessary to assign a name to this object. When we speak of constructive objects, their names can always be made in the form of natural numbers: the numbers of these objects in any kind of system of enumeration. In the theory of enumerable sets one can therefore confine oneself to a study of the natural series of numbers and its s-dimensional generalizations. If the concept of a computable function (which has natural numbers both for its arguments and its values) is assumed to be

1. V. A. Uspenskiy. General Definition of the Algorithmic Computability and Algorithmic Reducibility. Diploma paper, 1952.

already specified and refined (in the sense, for example, of the algorithmic or partially-recursive function), then the enumerable subset of a natural series can be defined as a set of values of a computable function. In general, an arithmetic function is computable when and only when its graph is an enumerable set.

The theory of enumerable sets includes thus, the theory of computable functions, meaning also the theory of algorithms. Furthermore, this theory covers also the constructive operation with non-constructive objects, with which we deal, for example, in models of formal deductive theories, which are constructed on the basis of a narrow calculus of predicates: the set of proved premises of such theories is enumerable. In general, if the names (which are usually constructive objects) of certain (even though actually infinite) sets are effectively constructed successively from names of certain initial sets (the latter can also actually be infinite), then the set of names obtained thereby will be (in contents) and enumerable set (in the refinement of the method of selection of the names and the meanings with which the "effectiveness" of construction is meant, one can say of a set of names which is already enumerable in the corresponding exact meaning.) The choice of names is realized here by specifying the method of numbering (a particular role in numbering is played by so-called Goedel numberings). The question naturally arises of how to choose the names of the objects in such a manner, so that the relations between the names reflect the relations between the objects designated by the names, and to what extent is this in general (without changing the names) possible. Of particular interest is this problem in those cases when the names pertain to non-constructive objects (and the transition from the object to its name denotes a certain constructivization of the object). Essentially all problems of the foundation of mathematics are connected with problems of this kind. D. Hilbert hoped that it is possible to have such a method of transition from non-constructive (actually infinite) objects to constructive "names" that denote these objects, for which the relations between the names will reflect all the properties (and relations) of the objects

themselves. As is known, these hopes by Hilbert were found to be unfounded. What is the matter? The works by V. A. Uspenskiy [6 -- 9, 12, 14][1] on the theory of enumerable sets and computable operations on sets are devoted to a circle of problems connected with questions of this kind and we shall proceed now to a discussion of some of them.

2. We shall begin the survey of these works by V. A. Uspenskiy with problems pertaining to the general treatment of methods of numeration.[2]. We Already mentioned the significance that Goedel numeration of certain sets of objects has in mathematical logic and in mathematics. The essence of this measure consists of arithmetization realized with its means: in that by means of it problems concerning the properties of certain objects are converted into arithmetical problems concerning the properties of numbers, which are names of these objects (in the given numeration). It is clear, however, that not every numeration can permit a deduction of any properties (or relations of the objects themselves from relations between the names of the objects. what then is the secret of the success of the Goedel numeration? In papers [7, 9, 12, 14] V. A. Uspenskiy[1] answers this question. He introduces the concept of the computable numeration of a system of sets (a particular case of which is the computable numeration of a system of functions) and covering numeration (in a certain sense containing in itself any other numeration) and explains that the presence of both of these properties in Goedel numerations indeed is the cause of the important role of the latter in the theory of computable functions and enumerable sets.

From among the number of concrete results, pertaining to the concept of computable numeration, let us mention the theorem proved by V. A. Uspenskiy already in 1951 (see [14], p. 160), that a system of all infinite

1. See also V. A. Uspenskiy, on operations over enumerable sets, dissertation, 1955.
2. The enumeration of a set M is called an arbitrary reflection α of an arbitrary set E of natural numbers on M.

enumerable sets of natural numbers does not admit not only a Goedel,[1] but in general no computable numeration [7, 14][2] and that the situation is the same for a system of all infinite resolvable sets of natural numbers.

3. The idea of an abstract study of numerations, in the development of which V. A. Uspenskiy engaged under the influence of A. N. Kolmogorov, has led V. A. Uspenskiy to introduce into the theory of computable functions (we recall that a computable function is considered as a graph, is a particular case of an enumerable set) a general concept of a program of function (and a method of programming of computable functions), independently of any particular refinement of the concept of the algorithms [7, 9, 12].[2] The program as a record in a certain code of a set of rules, defining the algorithm, is naturally related both with the given algorithm and with the choice of the code. The refinement of the concept of a program appears to be, therefore, at first glance to be dependent on the refinement of the concept of the algorithm. But any algorithm specifies a certain computable function (defined on the set of these objects to which it is applicable. On the other hand the concept of computable functions is independent of the method of refinement of the concept of the algorithm: any of the known refinements of the concept of the algorithm leads to the same class of computable functions. Independent of the refinement of the concept of the algorithm, the concept of its program can therefore be computed and obtained starting with the concept of the computable function. It is precisely in this way that V. A. Uspenskiy proceeds.

Inasmuch as any program is a word in a certain alphabet, and words in a given alphabet can be effectively renumbered with natural numbers, then without loss of generality one can assume that programs are natural numbers,

---

1. The non-enumerability of a set of Goedel numbers of all infinite enumerable sets was proved in 1953 by Reis.
2. See also V. A. Uspenskiy, on operation on enumerable sets, dissertation, 1955.

and each method of programming (method of formalization of a set of rules, defining the computable functions) is a certain numeration of a system of computable functions. We deal consequently with the fact that from among all the possible numerations of this system we want to separate those, which appear to be methods of programming. Introducing the concept of potentially computable and fully covering numerations V. A. Uspenskiy defines the method of programming as a numeration that is simultaneously potentially computable and fully covering, and he gives convincing arguments [12] in favor of the fact, that any method of programming is actually a numeration of this kind. Under a "program" of a function (for a given method of programming) is meant in this case the number of that function in the numeration, representing the given method of programming.

4. In connection with conditional algorithms[1]. or, as we shall call them sometimes, algorithms of reducibility, which are of particular interest for the theory of enumerable sets and problems of reducibility, great significance attaches to computable (in other words -- partially recursive) operators, defined on the system of all arithmetic functions, and converting the sets of arithmetic functions into arithmetic functions.

To any computable operator there corresponds an algorithm (conditional), which permits (if for example we deal with a computable operator F, which convers a single-phase arithmetic function ϕ into a single-place arithmetic φ), in accordance with the defined program, to establish for each x those values of the argument of the function ϕ , for which information on the function $\varphi(x)$ may be suitable for the calculation of ϕ , and to calculate $\varphi(x)$ if this information is accessible (i.e., if the machine (algorithm) has at its disposal equations of the type $\phi(x_i)=y_i$, where x_i and y_i are the recorded necessary values of x_i and $\phi(x_i)$. Concerning the computable operator F1 one can therefore say that, if <u>all</u> the information on the function ϕ (all the equations of the type $\phi(x)=y$ are avilable, it makes possible to derive all the information concerning the

1. Concerning these, see Section 6, Item 2.

function φ, (all the equations of the type $\varphi(x)=y$). Any computable function will therefore be converted by this operator into a computable function. But computable functions can be characterized by their programs, i.e., by numbers in a certain numeration. It is natural to expect therefore that each computable operator induces a certain computable function, which relates the numbers (programs) of the functions that they can process with the numbers (programs) of the results of the processing in the case when the processed functions are computable. The operator defines for computable functions and which converts the latter into computable functions is called by V. A. Uspenskiy [12] a constructive operator, if it induces the foregoing computable function, relating the programs of the functions. As expected, any operator defined for computable functions and continued to a computable operator (defined for all arithmetic functions) is constructive[1] [9, 7, 12]. For the particular case of Goedel numbers this theorem was known before.[2] It was found that the inverse was also true. Indeed, any constructive operator is continued to a computable operator. The "program" of a function (in the sense of V. A. Uspenskiy) is thus found to be so good a name for it that it yields, from the effectiveness of construction of such a name of function f from names of other functions, conclusions concerning the effectiveness of the calculation of f on the basis of information concerning these other functions.

A theorem analogous to that of V. A. Uspenskiy, for the particular case of operators which convert general-recursive functions into a general-recursive ones, and specially Goedel numerations of functions, was proved by G. S. Tseytin [7].

At the same time, even with the aid of so good a name for a computable function as its "program" (in the

1. V. A. Uspenskiy. On Operations Over Enumerable Sets. Dissertation, 1955.
2. See S. C. Kleene, Introduction into Meta-Mathematics, Moscow, Foreign Literature Press, 1957 (henceforth frequently cited as "Kleene (Meta-mathematics)), p. 308, theorem XXIV (c).

sense of V. A. Uspenskiy), it is impossible to represent fully the properties of the function itself: there exists no algorithm which permits to establish from the program of the function whether this function has a given property or not (A more accurate a general formulation of this theorem can be found in the papers [7] and [12] of V. A. Uspenskiy).

5. All the above-mentioned results and concepts, pertaining to the general theory of numerations, programs, and computable operators, were obtained by V. A. Uspenskiy in the general theory he constructed for systems of enumerable sets and computable operations on sets[1]. [6, 7]. (In particular, a computable operator is a particular case of a computable operation on sets -- on graphs of functions.)

In the definition he proposed for a computable operation, V. A. Uspenskiy undertook to cover all types of such operations, which convert successively (with admission of a transition to the limit) starting with finite subsets (Corteges), enumerable sets or aggregates of sets in enumerable sets. As shown by V. A. Uspenskiy [6], for these operations it becomes possible to have also certain other definitions, based on the concept of operations of A. N. Kolmogorov and operations of Post.[2]. But the definition of V. A. Uspenskiy is convenient because it exhibits a connection between the computable operations and the continuous representations, and therefore can serve as an instrument for constructivization of mathematical analysis (see Section 10).

V. A. Uspenskiy considers systems of enumerable sets as topological spaces. Although these spaces are found to be sufficiently "poor" T_0 spaces, V. A. Uspenskiy succeeds, by introducing the concept of τ-density, to transfer to them, mutatis mutandis (when the term "everywhere dense" is replaced by "τ-dense"), the theorem that two continuous

1. V. A. Uspenskiy, On Operations Over Enumerable Sets, Dissertation, 1955.
2. E. L. Post, American Journal of Mathematics, 65 (1943) 197 and Bulletin of American Mathematical Society, 54, No. 7 (1948), 641.

representations of a topological space X into a topological space Y coinciding on a certain every-where dense subset of the space X, coincide also on all X. With the aid of this theorem he indeed proves the assumptions given in item 3 concerning computable and constructive operators (a computable operator is a special type of continuous representation of the function space) [6, 12]. The theorem mentioned in the same item 3, concerning the indistinguishability of the properties of a function from its program, is a simple consequence of the topological connectivity of the system of computable functions [12].

6. Definitions of the Mass Problem and of Algorithmic Reducibility of Mass Problems. Structure of Degrees of Difficulty.

1. Algorithms are sought in mathematics as general methods of effective solution of any (single) problem of a definite kind. The broader the class of problems solved by means of a certain algorithm, the greater usually the value of this algorithm. In searches for a general and effective method of solving broader and broader and more varied classes of problems, lies, essentially, the history of mathematics. The need for a general definition and a general theory of algorithms has also arisen in connection with searches for algorithms for a solution of certain classes of problems, the stubborn lack of success in which (these searches) should give rise to suspicion concerning the general non-realizability of the sought algorithm. Thus, in connection with an algorithm one naturally considers the class of problems which it admittedly can solve. Since any finite number of single problems can always be considered as a (more complex) single problem, then interest attaches only to the case of an infinite class of problems. In this case, however, one can also speak of a single -- mass -- problem, the solution of which should indeed consist of finding the algorithm.

The first definition of a mass problem connected with the refinement of the statement of the problem of existence

or non-existence of an algorithm, which would recognize the presence or absence of a certain property in objects of a definite kind, was proposed by A. A. Markov. He ([48], Chapter V) gives the name of "mass problems" to problems of the following type.

We consider a certain class of single problems, each of which is a problem that requires a positive or negative answer. The problem is raised of finding a single general constructive method of finding the correct solution for any single problem of the considered class. This formulation is refined later on by replacing the term "single general constructive method" by the term "algorithm." A requirement is imposed in this case on the algorithm, that it be applicable for the recording of any single problem of the considerec class[1] and that it convert this recording into a word "yes" (or into some analogue of this word, for example, an empty word). If the problem is solved in the positive sense, and into the word "no" (or some of its analogues, say a non-empty word), when it is solved in the negative meaning. The use of the principle of normalization of algorithms permits an even greater refinement of this formulation of the problem, by formulating a definition of a mass problem for a given class of single problems as the problem of constructing a normal algorithm on an alphabe of recordings of considered single problems, on an alphabet of recordings of the considered single problems, which converts the recording of the single problem into an empty word if and only if this problem is solved in the positive sense. The mass problems so stated are called by A. A. Markov normal mass problems.

The problem consisting of searching an algorithm, which would permit for any given object of a definite kind to solve the problem whether this object has a given property S (whether it belongs to a given set M) or not, is called usually the problem of solvability. Mass problems in the sense of A. A. Markov are thus problems of solvability. In the more general sense, the mass problem is defined (together with its solution) by a student of

1. That is, it would be a non-partial algorithm.

A. N. Kolmogorv, Yu. T. Medvedev [5, 6][1].

With any solvability problem, P there is an associated function f -- a characteristic function of the set M of objects, having the property of interest to us in the particular problem. Medvedev considers the case when (with the aid, for example, of arithmetization methods) this function can be assumed to be arithmetic and furthermore defined over the entire natural series of numbers (recall that the mass problem is formulated for an infinite class of objects). The solvability of the problem P is naturally defined in this case with the general-recursiveness of the function f.

Matters are somewhat different in the case of the so-called problems of enumerability. The enumerability problem of a set M is called the problem of relating to each number n an element of the set M, i.e., to construct a function f of natural numbers, the values of which would be the names (and furthermore all of them) of the elements of the set M. If one uses as such names natural numbers, then the sought function f will again be arithmetic (numerical), defined over the entire natural series. But if the set M contains more than one element, then the conditions of the problem are known to be satisfied by an infinite set of functions (defined everywhere) f. If one refines the statement of the problem and requires an effective method for solving the problem, then it is natural to consider the problem of enumerability of the set M to be solvable, if in the corresponding class of function f corresponding to it there is at least one general-recursive function.

A somewhat more unique situation prevails for the so-called problems of separability. Under the separability problem of two sets P and Q (we shall presuppose right away that they are sets of natural numbers) one has in mind a problem, the statement of which can be refined, for example, as follows: find an arithmetic function f, defined over the entire natural series (the natural series we denote

---

1. See also Yu. T. Medvedev, Degrees of Difficulties of Mass Problems, Self-Abstract of Dissertation, Moscow, 1955.

henceforth by the letter N), which would satisfy the following requirements: a) if $n \in P$, then f(n) = 1; b) if $n \in Q$, then f(n) = 0. It is clear that if the sets P and Q do not have common elements, but are complements of each other, then the class A of the function satisfying the conditions of the problem contains one unique function; if $P \cap Q = 0$, but $P \cup Q \neq N$, then the Class A is infinite, but if $P \cap Q \neq 0$, it is empty. A further refinement of the statement of the problem in connection with the requirement of effectiveness of its solution leads again to an identification of the solvability of the problem with the general-recursiveness of at least one of the functions $f \in A$.

One can cite also many other examples of various types of problems of the same kind, which exist - each separately - in mathematics, not simply as a class or series of unique problems, but as one special problem of finding an everywhere-defined arithmetic function, satisfying required conditions, whereby the effectiveness of solution of this problem is understood in the sense of the general-recursiveness of the sought function. It was precisely this kind of problem that Yu. T. Medvedev proposed to call <u>mass problems</u>.

Inasmuch as a mass problem in the sense of Yu. T. Medvedev always corresponds to a class $\{f\}$ of functions which are its "solutions" and, to the contrary, each class A of functions of a natural argument with natural values values defines a mass problem: constructive function $f \in A$, then Yu. T. Medvedev [5, 6][1] identifies the mass problem with a class of functions, defined over the entire natural series, the values of which are also natural numbers. The problem $A = \{f\}$ is called solvable if there exists a general-recursive function $f \in A$, and unsolvable in the opposite case. The problem A is called proper if the class A is empty.

2. The algorithms constructed by mathematicians to solve entire classes of functions (or corresponding mass problems corresponding to such classes) do not have by far always an absolute character. Just as the solution

1. See also Yu. T. Medvedev, Degrees of Difficulty of Mass Problems, Abstract of Dissertation, Moscow, 1955.

of a unique problem (for example, construction problem) is sometimes taken to mean its reduction to a finite number of problems, accepted as a solution, so is a solution of a mass problem consist frequently of an effective reduction of this problem to another (or to other) mass problems. (Thus, for example, the problem of differentiation of a product of two functions f and g reduces effectively the problem of differentiation of functions f and the function g. A large number of algorithms of mathematics, bear, essentially, indeed this character).

How can one refine, however, the concept of effective reducibility of a mass problem A to a mass problem B?

Many definitions of the effective reducibility for problems of computability of functions and solvability of predicates (or sets corresponding to them) were proposed by Post and Kleene.[1] One of the known ones among these is the definition introduced by Post[2] for Turing reducibility (in terms of a machine-specified process). The Soviet mathematicians have proposed many definitions for effective (algorithmic) reducibility in various meanings. The mutual relationships have been clarified by V. A. Uspenskiy [10].

Chronologically, the first of the formulated definitions advanced by Soviet mathematicians is apparently the one which is close to that of Kleene, for the reducibility of functions, the idea of which belongs to B. A. Trakhtenbrot. V. A.[3] explains this idea as follows: the function $\gamma(n)$ reduces recursively to the function $\delta(n)$, if $\gamma(n)$ belongs to the recursive closure $\delta(n)$. (The recursive closure of the function δ is the minimum recursively-closed class, containing δ and all the primitive-recursive functions. The class of functions is called recursively-closed if it is closed relative to recursive operations: superposition, primitive recursions, and the application of the operator μ : the smallest

1. See S. C. Kleene, Introduction to Meta-Methamatics, Moscow, Foreign Literature Press, 1957, pp. 280 -- 281.
2. E. L. Post. Recursively enumerable sets of positive integers and their decision problems. Bulletin, American Mathematical Society, 7, No. 5 (1944). Henceforth designated "Post (1944)."

such that...)

Already in his student paper[1] V. A. Uspenskiy proposed a new definition of the effective reducibility in terms of the algorithm of A. N. Kolmogorov. The algorithm of A. N. Kolmogorov can be imagined as realizable by means of a machine, which converts topological complexes of a definite type (Π - complexes) into complexes of the same type. In accordance with this, the problem and its solution are also given in the form of Π-complexes. One can imagine also in the form of Π-complexes the initial state of the machine (the machine starts operating when one joins to the initial state, which has the form of a Π-complex, the input date in the form of another Π-complex).

The algorithm of Kolmogorov is called by V. A. Uspenskiy[1] unconditional, if the initial Π -complex is empty, and conditional in the opposite case. Since a conditional algorithm with a finite initial complex can always be replaced by an equivalent unconditional one, then only conditional algorithms with finite (but limited[2]) initial state are of interest. In terms of such conditional algorithms, V. A. Uspenskiy[1] [3] indeed defines the algorithmic reducibility of the functions.

Inasmuch as (see Section 4, item 5) the numerical functions $\gamma(n)$ and $\delta(n)$ correspond to functions of complexes $\Gamma(K)$ and $\Delta(K)$, the problem of algorithmic reducibility of numerical functions reduces to the problem of algorithmic reducibility of functions of complexes. It is therefore enough to define this latter reducibility, which is indeed done by V. A. Uspenskiy. His idea of the definition consists of the following [3].

One considers the reducibility of a function of complexes Γ to a function of complexes Δ . An infinite

---

Footnote (3) from pg. 68. V. A. Uspenskiy. General Definition of Algorithmic Computability and Algorithmic Reducibility. Diploma paper, 1952.

1. Ibid.

2. The following are limited: the characteristic function that relates the number to each vertex of the complex, and the aggregate set of symbols at the ends of segments departing from the verticies.

complex U_Δ is constructed, which includes in itself all the information concerning the function Δ. By $K \circ U_\Delta$ we denote the complex that arises as a result of a joining of the complex K to U_Δ, realized by some special method. The function Γ is called algorithmically reducible to the function Δ, if there exists such an algorithm (in the sense of A. N. Kolmogorov) ϕ, that for any complex K,

$$\Gamma(K) = \phi(K \circ U_\Delta).$$

Just as any computable (i.e., partially recursive) function admits of a canonical representation in terms of primitive recrusive functions and the operator μ (see Section 3), so does there exist for each function $\gamma(n)$, which is algorithmically reducible to a function $\delta(n)$, a canonical representation in terms of the function $\delta(n)$, certain primitive-recursive functions, and the operator . One of these representations was obtained by V. A. Uspenskiy on the basis of the theorem which he proved in his diploma paper and in paper [3]: Let the function $\gamma(n)$ be algorithmically reducible to everywhere-defined function $\delta(n)$. Then there exists a primitive-recursive function $h(x, y, z)$ such that if the function $g(n, m)$ is specified by the recursions

$$g(n, 0) = n,$$

$$g(n, m+1) = h(m, g(n, m), \delta(m)),$$

then the function $\gamma(n)$ will be calculated from the formula

$$\gamma(n) = \tau(g(n, \mu m[\omega(g(n, m)) = 0])),$$

where $\tau(x)$ and $\omega(x)$ are fully defined forever-fixed recursive functions. From this representation it is evident, in particular, that the function γ reduces algorithmically to the function δ, if it belongs to the recursive closure δ, and since the inverse theorem is obvious, so is also obvious the equivalence of the algorithmic and recursive reducibility (the equivalence of the recursive and Turing reducibility is also established in the diploma paper of 1952).

3. All the heretofore considered definitions of reducibility are found thus to be definitions of the reducibility of functions relative to the computability. (Since

the reducibility of the predicates (or sets) by solvability can be considered as reducibility of the characteristic functions by computability, these definitions can be used also in application to the reducibility by solvability.) A new type of reducibility -- reducibility by enumerability -- was introduced into consideration by V. A. Uspenskiy[1.] [6, 10].

The definition of reducibility by enumerability proposed by V. A. Uspenskiy is based on the concept he introduces of the computable operation on sets, which was already discussed in the preceding section. With the aid of this concept, the reducibility by enumerability is defined as follows[1.] [6, 10]: a set R is called reducible by enumerability to sets $S_1, \ldots, S_l$, if there exists such a computable operation U, that $R = U(S_1, \ldots, S_l)$.

In terms of reducibility by enumerability one can express reducibility by computability (meaning also reducibility by solvability). In fact (V. A. Uspenskiy [10]: a function φ reduces by computability to the functions $\psi_1, \ldots, \psi_l$ when and only when the graph of the function φ reduces by enumerability to the graphs of the functions $\psi_1, \ldots, \psi_l$. In turn, reducibility by enumerability can be expressed in terms of reducibility by computability. For this purpose it is sufficient to introduce into consideration the concept of proper function of the set M, which equals to 1 when $m \in M$, and which is not defined for $m \bar{\in} M$. Actually, the set R reduces by computability to the sets $S_1, \ldots, S_l$, when and only when the proper function of the set R reduces by computability to the proper functions of the sets $S_1, \ldots, S_l$.

The most general definition of reducibility as a relation, of which one can speak as applied to any mass problem considered as classes (or families) of arithmetic functions (see item 2) (including also the problems of computability, solvability, and enumerability), was proposed by Yu. T. Medvedev[2.] [5, 6]. According to Yu. T. Medvedev, a mass problem A reduces algorithmically (or

1. V. A. Uspenskiy. On Operations on Enumerable Sets. Dissertation, Moscow, 1955.
2. Yu. T. Medvedev. Degrees of Difficulties of Mass Problems. Dissertation, Moscow State University, 1955.

simply reduces) the mass problem B, if there exists a partially-recursive operator $\mathfrak{R}$[1]; applicable to any function $g \in B$ and reducing it to a certain function (a function which depends on g).

4. The basic distinguishing feature of Medvedev's definition is that in its terms the problem of reducibility of one mass problem to another (or to others) can in turn be considered as a certain mass problem (problem of reducibility). How this is done, will become clear from what follows.

Related to each definition of reducibility of A to B is a relation of the equivalence type, which is tantamount to saying that A reduces to B, and, conversely, B reduces to A. Such a relation breaks down the object region, to which A and B, into classes, which do not have common elements, which we shall call degrees (relative to a given reducibility relation) and denote with small latin letters. Since any relation of reducibility is reflected intransitive, a set of degrees coressponding to it can be imagined to be partially ordered with the aid of this relation: if A reduces to B, we shall say that $a < b$, where a and b are the degrees to which A and B belong respectively.

For reducibility by solvability, computability, or enumerability we obtain thus the following sets: degrees of non-solvability (P), degrees of non-computability (Ч), degrees of non-enumerability (Π). According to Yu. T. Medvedev, there corresponds to the reducibility of mass problems a set of degrees of difficulty (Ω).

A survey of the construction of these sets and of the relations between them is contained in the paper by V. A. Uspenskiy [10]. Thus, in each of partially-ordered sets P, Ч, Π and Ω there is the smallest element: In P -- the class ("degree") of solvable predicates. In Ч -- the class of computable functions, in Π -- the class of enumerable sets, and in Ω -- the class of solvable mass problems. Each of the sets P, Ч, Π and Ω

1. The definition given by Kleene for a partially recursive operator can be found, for example, in the book by S. C. Kleene, Introduction to Meta-Mathematics, Moscow, Foreign Literature Press, 1957, p. 291.

is the upper semi-lattice (i.e., any of its elements have the least upper edge). The semi-lattices **P**, **Ч**, **П** have a cardinality $c = 2^{\aleph_0}$ and do not contain maximal elements. The semi-lattice Ω has a cardinality 2^c and contains a largest element (degree of difficulty of the improper problem, i.e., of an empty family of functions). The set of degrees of non-computability (**Ч**) and degrees of non-enumerability (**П**) are isomorphic [6, 10], the set of degrees of non-sovability (P) is isomorphic only to part (**Ч'**) of the set of degrees of non-computability (**Ч**), namely to the set of degrees of non-computability of everywhere-defined functions. In turn, the set of degrees of non-enumerability is isomorphic to the regular part (Π_0) of the set of degrees of difficulty (Ω)[1].

The semi-lattice P, as shown by Kleene and Post, is not a lattice. The semi-lattice Ω as shown by Yu. T. Medvedev, is a lattice and even an implicative lattice, which can be interpreted as a logic. The least upper edge of two degrees of difficulty corresponds in this case to a conjunction, and the largest lower edge corresponds to a disjunction. The implication of two mass problems A and B (more accurately, the degrees of difficulty a and b corresponding to them) can be understood as a problem C of the least degree of difficulty c such that $a \wedge c \geq b$ ($\wedge$ is the conjunction sign). The implicativity of the lattice Ω indeed consists of the fact that such a c exists for any a and b. For this c it is natural to introduce, as done by Yu. T. Medvedev, the notation $a \supset b$. If one identifies the mass problem with the degree of its difficulty, it is found that the mass problem $a \supset b$ is solvable if and only if $a \geq b$, i.e., when b reduces to a (in the sense of Yu. T. Medvedev). It is therefore natural to interpret $a \supset b$ as a problem in reducibility (b to a).

The problem of convergence of the reduction of the mass problem A to the mass problem B is, therefore, in itself a mass problem, unsolvable in the case when A does not reduce to B. We shall find this remark useful in the

1. S. C. Kleene and E. L. Post. The Upper Semi-Lattice of Degrees of Recursive Unsolvabilities, Annals of Mathematics, Series 2, 59, (1954), 379 -- 407.

next section, in connection with Post's problem of reducibility).

As it is known, in an implicative lattice one can always introduce negation through implication. However, since the improper problem reduces only to an improper one, then the introduction of the negation of a as $a \supset \infty$ (where ∞ is the degree of difficulty of the improper problem) would identify negation of any problem different from the improper one, with the improper problem, i.e., it would not give the natural meaning of negation. Yu. T. Medvedev avoids this by going over from the lattice Ω to its segment $0 \leq x \leq e$, where 0 is the least degree of difficulty and e is an arbitrary degree of difficulty, different from ∞. Inasmuch as any such segment of an implicative lattice is in its turn an implicative lattice, then, defining the negation of a as $a \supset e$, Yu. T. Medvedev obtains[1] [5, 6] an interpretation of the "intuitionistic" logic of formulations, the relations of which to the calculus of A. N. Kolmogorov's problems (1932) will be treated in Section 11.

5. For the structure of degrees of difficulties Yu. T. Medvedev[1] [5, 6] established next a series of theorems, pertaining to the problem of the existence of mass problems with certain particular properties. Thus, for any degree $a \neq \infty$ there exists a problem of solvability, the degree of which is higher than a (i.e., each improper interval (a, ∞) of the lattice Ω is not empty). To the contrary, for each degree a of any particular problem in solvability there exists a least x such that $x > a$ (i.e., the interval (a, x) is empty). The simplest example[1] of an empty interval (a, b) is obtained by putting $a = 0, b = z$, where z is the problem of constructing a function which is not general-recursive (such a problem is obviously the problem of the least degree of difficulty in a class of unsolvable problems). If the degree $a \neq 0$ is obtained from the degrees of the problem of solvability with the aid of the disjunction operation with the aid of the operations of disjuction, conjunction, and implication,

1. Yu. T. Medvedev, Degrees of Difficulty of Mass Problems, Dissertation, Moscow State University, 1955.

then there exists a non-vanishing degree $b<a$. A theorem is proved[1] [5] with respect to problems of enumerability and solvability.

An investigation of the calculus of mass problems, created by Yu. T. Medvedev, was continued further by A. A. Muchnik.

A. A. Muchnik considered, in particular, the problem A_ϕ -- continuability of a partially-recursive function $\phi(n)$, consisting of all the functions coinciding with $\phi(n)$ at points where $\phi(n)$ is defined. It was found here ([1], theorem 4) that if the problem B consists of one function and reduces A_ϕ, then B is a solvable problem. As a consequence of this we obtain immediately that if the problem of solvability of a set E reduces to some sort of problem of separability of an enumerable set, then the set E is solvable.

The following theorems hold:

Theorem 5. For any pair of recursively unseparable enumerable sets E_1 and E_2 there exists an enumerable unsolvable set H such that the problem of separability of E_1 and E_2 does not reduce to the problem of solvability of H.

Theorem 6. There exists an enumerable sequence of pairwise uncomparable problems of separability of enumerable sets.

Theorem 7. For any unsolvable $A_{E_1E_2}$ of separability of enumerable E_1 and E_2 there exists a problem of separability $A_{H_1H_2}$ of recursively unseparable enumerable sets H_1 and H_2 such that $A_{E_1E_2}$ does not reduce to $A_{H_1H_2}$.

Among the results of A. A. Muchnik, pertaining to the calculus of mass problems of Yu. T. Medvedev, belongs above all his solution of Post's reducibility problem, on which we shall dwell especially in the next section.

1. Yu. T. Medvedev, Degrees of Difficulty of Mass Problems, Dissertation, Moscow State University, 1955.

7. Post's Reducibility Problem and Problems Related With It.

1. The theory of enumerable sets is of interest in connection with problems of solvability. Historically, this term was first applied to problems of solvability of logical calculi, under which were meant problems of construction of an always applicable (i.e., not partial) algorithm, which permits, in accordance with the form of the formula of a certain calculus, to solve the problem whether it is provable in this calculus or not. As a consequence, problems of solvability (see Section 6) of the set M have begun to assume the meaning of any mass problem consisting of the construction of a calculable function, which assumes, for example, a value 1 if the object belongs to M and 0 if it does not. (It is clear that instead of "the belonging of the objects to the set M" one can speak of "its having the property S which defines M.")

As already noted, it is most natural to imagine an enumerable set as a set M of constructive objects, successively generated one after the other in some regular process. If this process continues without limit, then the problem of the belonging to an arbitrary object a to a set M is solved effectively (in the sense of the existence of an algorithm which recognizes whether a belongs to the set M or not) only under the condition that not only the set M itself is enumerable, but also its complement. If the complement to the enumerable set M is not enumerable, then the set M is unsolvable.

With the aid of the diagonal procedure it is easy to construct an enumerable but unsolvable set thereby obtaining an example of an algorithmically unsolvable problem of solvability. And then the solution of the problem of the non-solvability of any mass problem of solvability P can be sought already along the ways of algorithmic reduction to the problem P of another algorithmic problem Q, the unsolvability of which has already been proved. It was precisely in this manner that the most important results were obtained, pertaining to the unsolvability of mass problems, beginning with the problems of solvability of a narrow calculus of

predicates (in this case also the results, considered in Section 9, of A. A. Markov, P. S. Novikov, S. A. Adyan, G. S. Tseytin, and others). In all these cases the problem Q (which they reduced to the considered problem P) can be chosed to be in the same problem which plays the role of a standard, for example, the problem of solvability of the predicate $(Ex)T_1(a, a, x)$ (or the corresponding enumerable set: $\hat{z}(Ex)T_1(z, z, x)$[1]). This predicate has the highest degree of unsolvability compared with any predicate of the type $(Ex)R(a, x)$, where R is a recursive two-place predicate (a predicate of the type $(Ex)\ R(q, x)$, and can define, as is known, any enumerable set). In particular, any enumerable but not solvable set reduces by solvability to a set T: $\hat{z}(Ex)T_1(z, z, x)$. The question arises naturally whether the reverse is true: do there exist enumerable but unsolvable sets with a smaller degree of unsolvability than in T? Can one, in other words, reduce (by solvability) T to any enumerable but not solvable set? This problem was raised in 1944 by Post and is known as Post's reducibility problem. For its solution, naturally, it was found convenient to analyze the construction of the set of all enumerable sets and analyze all possible means of reducing (by solvability) certain sets to others. In the foreign literature very many papers devoted to this problem have been published, most important of which are those written by Post, Kleene, Turing, Reis, Decker, Spector, Friedberg and others. From among the Soviet mathematicians, at the initiative of P. S. Novikov and A. N. Kolmogorov, who attracted to these problems the attention of the students, were engaged B. A. Trakhtenbrot, A. V. Kuznetsov, V. A. Uspenskiy, Yu. T. Medvedev, and A. A. Muchnik. The Post problem was solved almost simultaneously and almost independently of each other by the young mathematician A. A. Muchnik and the American mathematician Friedberg (certain additional results obtained by Muchnik are lacking in Friedberg's work). But the works preceeding the results of Muchnik and Friedberg and pertaining to problems of classification of enumerable sets, construction of their system, and

1. See S. C. Kleene, Introduction to Meta-Mathematics, Moscow, Foreign Literature Press, 1957, p. 253.

the interrelation of various types of enumerable operators, retain their significance independent of the Post problem.

2. The problem of reducibility, which was formulated by Post in connection with the general premise, created in the region of problems of solvability of formal logical-mathematical calculations, was found to be, as was noted for example by Myhill[1], connected in a certain sense with the famous theorem of Goedel concerning the incompleteness of formalized systems, containing arithmetic. A set of provable formulas of such systems is enumerable, but not solvable, i.e., their complement is not enumerable. Furthermore, for any formal system Σ (with effective rules of deduction), permitting to record in its terms all equalities of the form $f(x) = y$, where f is a computable (i.e., specified by some algorithm) numerical function such that the equalities $f(x) = y$ are provable by the means of the Σ system if and only if they are contentfully true, the following situation takes place: a) the set D of Goedel numbers of provable formulas of such a system is enumerable; b) the complement D' to the set D is not enumerable; c) among the subsets of set D' there is an infinite enumerable set; d) among the subsets of set D' there is no "maximal" infinite enumerable set R ("maximal" in that sense, that the difference $M \setminus R$ no longer contains an infinite enumerable set); e) there exists a computable (and even a general-recursive) "reproducing" function p such that if n is a Goedel number of an infinite enumerable set $P \subseteq D'$, then the number $p(n)$ belongs to the difference $D' \setminus P$.

In other words, the set D^* of the provable formulas of the system Σ (we note that this sytem is non-contradictory, since not all of its formulas are provable) such that one can add to it any infinite enumerable set P^*_1, of formulas that are not contained in them, and nevertheless one finds effectively after this a formula which is not contained in the junction $D^* \cup P^*_1$[2]. To this

1. J. Myhill, Creative Sets, Zeitschr. Mathem Logik u. Grundlagen Math. 1, (1955), 97 -- 108.
2. The asterisks are used here because we have changed over from Goedel numbers of formulas to the formulas themselves.

union one can add again an infinite enumerable set P^*_2 of formulas not contained in it and again, from the Goedel number of this set, one can effectively indicate at a formula not contained in the union $D^* \cup P^*_1 \cup P^*_2$, etc.[1])

The sets of natural numbers with properties a) -- e), were called by Post creative. The enumerable set, having the property b), but not having the property c) (i.e., such that its complement is infinite, but does not contain an infinite enumerable subset), he called simple. Among the enumerable sets the creative ones (all of them!) have one in the same, higher, degree of unsolvability: any enumerable set is reduced to them by solvability. (The "standard" T, which we discussed in item 1, is consequently such a creative set.) But is a creative set, for example, reduced to any simple one? The Post problem would be solved were it possible to answer this question in the negative. But do simple sets exist at all? And how can one prove the unsolvability of such a set without resorting to reducing the standard to it? Already in 1944 Post succeeded in constructing an example of a simple set P, the proof of the unsolvability of which did not consist of reducing (of the proposed) standard T to P. However, the question of whether T nevertheless reduces to a simple set constructed by it was answered in the affirmative: "It reduces." The "simplicity" of the set by itself was thus found to be insufficient in order for T not to be reduced to it. Then Post

1. As shown by V. A. Uspenskiy [14] in item e) it is possible to replace the enumerable function p, which reproduces the number p(n), belonging to the difference $D' \setminus P$, by means of a computable function g, which reproduces the Goedel number g(n) of a certain infinite enumerable set $Q \subseteq D' \setminus P$, i.e., one can speak not of an effective searching for a formula not contained in the union $D^* \cup P^*$, but of an effective definition of an infinite enumerable set of such formulas. In the same note [14], V. A. Uspenskiy constructed an example of an enumerable set, the complement of which has the properties b)—d), but for which e) is not true.

separated among the simple sets a particular class of hypersimple ones, (a hypersimple set is an enumerable set H, the complement of which H' is infinite and such that there exists no enumerable set or pairwise non-intersecting corteges, each of which intersect with H') and attempted to prove that the creative set does not reduce to the hypersimple one. By way of methods of reduction he used in this case a special apparatus (Post's tables), developed by him specially for the reducibility by solvability, i.e., pertaining to predicates ("arithmetic" functions, which assume one and only one of two values, 0 or 1). It was found that with the aid of this apparatus the creative set does not reduce to the hypersimple one. But the question remained open of whether this remains true also for the more general concept of algorithmic reducibility. To answer this question it was necessary to investigate above all various possible methods of reduction -- the mutual relationship between various classes of enumerable operators. This problem was especially investigated by B. A. Trakhtenbrot [3, 12], who was later joined by A. V. Kuznetsov [2].

3. In his papers [3, 12] Trakhtenbrot investigated the mutual relationship between the following types of enumerable operators: a) primitive-recursive[1]; b) those he called Post "operators"[2]; c) general-recursive[3]; d) partially-recursive.

1. The expressions: a) « $\varphi = T(\psi)$ », where T is a primitive-recursive operator, and b) " φ is uniformly primitively-recursive relative to ψ " denotes the same thing. See Kleene, Introduction to Meta-Mathematics, Moscow, Foreign Literature Press, 1957, p. 210 -- 211.
2. Post operators correspond to Post's tables. As established by B. A. Trakhtenbrot [3], a Post operator can be represented in the following form: we consider a primitive-recursive operator T, which processes a set of n functions $F_0, f_1, \ldots, f_{n-1}$ into the function f, and we place instead of f_0 a certain defined general-recursive function ψ. The obtained operator $T(\psi, f_1, \ldots, f_{n-1})$ is indeed a Post operator of n -1 functional variables.
3. A partially-recursive operator is called general-recursive if it transforms any everywhere-defined function into one everywhere-defined.

It was found that between these classes of operators there exist relations of strict inclusion, i.e., each successive class is broader than the preceding one. It was found further that a creative set does not reduce to a hyper-simple one not only by means of Post operators but even by general-recursive operators (E. A. Trakhtenbrot [3, 5], A. V. Kuznetsov [2]. Yet Decker observed that the situation is different in the case of partially recursive operators: for each creative set K there exists a hyper-simple one H, to which K reduces by means of a partially recursive operator[1]. This naturally did not exclude the possibility of their existing among the hyper-simple sets themselves partially-recursive operators that do not reduce one to another, i.e., that hyper-simple sets can have different degrees of unsolvability. The problem raised by Post for enumerable sets has shown thus an analogous problem for hyper-simple sets.

At the same time the following problems arose:
a) study of the structure of a system of hyper-simple sets;
b) clarification of the role of partially-recursive operators which are not general-recursive. The first of these was first engaged in by Yu. T. Medvedev [4], V. A. Uspenskiy [14], and A. V. Kuznetsov (concerning the latter, see [14], for example). Together with solving the Post problem, the problem was completely solved by A. A. Muchnik [1, 3][2]. The fact that operators which are partially

1. In fact, Decker has also shown that for each non-enumerable set E_1 there exists a hyper-simple set E_2, to which E_1 can be reduced by a partially recursive operator.
2. A detailed exposition of the results of A. A. Muchnik, pertaining to Post's problem (with complete proofs) is found in his dissertation "Solution of the Post Solvability Problem" (the abstract was sent out on 14 November 1958). The contents of his two papers at the Moscow Mathematical Society of 16 October 1956 and 17 December 1957 werepublished in two articles: "Solution of the Post Reducibility Problem and Certain Other Problems in the Theory of Algorithms. I" and "Isomorphism of Systems of Recursive-enumerable Sets with Effective Properties" (Trudy Moskovskogo Matematiches-

recursive but not general recursive can play a very substantial role as a reduction method was explained, in particular, by A. V. Kuznetsov [2] and B. A. Trakhtenbrot [5] in their joint paper. It is with this work that we now begin an examination of the second of the foregoing problems (item 5 is devoted especially to the first one).

4. Corresponding to the intuitive meaning of a "reduction" of the calculation of one function (ψ) to others $(\varphi_1, \ldots, \varphi_k)$ is essentially indeed the concept of the partially recursive (or computable) operator, which permits, given enough information on the values of the functions $\varphi_1, \ldots, \varphi_k$ to calculate the values of the functions ψ (if they are defined). A general-recursive operator (we shall confine ourselves for simplicity to the case k = 1) is also partially recursive, but satisfies the additional requirement that any every-where defined function is converted by it again into one everywhere-defined. It could happen, incidentally, that an operator which converts any one everywhere-defined function into one not everywhere defined, is perhaps not worthy of the name "computable operator."

Such a point of view was particularly tempting in connection with the results of V. A. Trakhtenbrot [3], which is shown that from a a hypersimple set[1] it is impossible to obtain a creative set by any general-recursive operator. (Were it possible to neglect partially-recursive operators in the statement of the Post problem, then the Post problem could be thus solved). During the course of a discussion that took place on the Seminar on Mathematical Logic at the Moscow State University in connection with the paper by B. A. Trakhtenbrot (November 1954), which contained a detailed proof of this result, arguments against this point of view were raised by V. A. Uspenskiy

---

Footnote (2) cont. from Pg. 81 ... kogo obshchestva [Works of the Moscow Mathematical Society] 7 (1958), pp. 391 -- 405 and 407 -- 412. His solution to the Post problem was reported by A. A. Muchnik to the participants of the Seminar on Mathematical Logic at the Moscow State University at the end of 1955.

1. The set is identified here with the corresponding characteristic function.

(published in [13], theorem 11) and A. V. Kuznetsov, who constructed examples of partially-recursive operators which are not general-recursive, and nevertheless reduce any general-recursive function into a general-recursive one. The value of partially-recursive but non-general recursive operators was particularly emphasized thereby in connection with considerations pertaining to the concept, proposed by A. V. Kuznetsov, of "general-definiteness" and "field of general-definiteness" of an operator. The participants of the seminars led by P. S. Novikov, and the students at his courses on the fundamentals of mathematics and on the descriptive theory of functions, usually consider (particularly since 1954) everywhere-defined single-placed numerical functions as point in the Baire space. This method of working with arithmetic function was illustrated in greater detail in the article by V. A. Uspenskiy [13]). The system of all single-placed numerical functions (including those not everywhere defined) is indentifiable, following V. A. Uspenskiy ([13], Section 10) with the generalized Baire space $\hat{J}$. The function F is called general-defined at the point $a \in \hat{J}$, if F is defined at the point a and $F(a) \in J$ (i.e., if the operator F converts a into an everywhere-defined function). The aggregate of all points of ordinary Baire space J (i.e., of all everywhere-defined single-placed numerical functions) in which the function F is general-defined, is called the region of general-definiteness of the function (operator) F.

Let γ be any everywhere-defined arithmetic function, the complement to the graph of which is an enumerable set. A. A. Muchnik has shown that the set $J \setminus \{\gamma\}$ (i.e., the set of all the points of ordinary Baire space, different from γ) is the region of general-definiteness of a certain partially-recursive operator. If in this case the function γ is not general-recursive, this proves in itself the existence of a partial-recursive operator, which although it is not general-recursive, nevertheless it not only transforms any general-recursive function into a general-recursive one, but in general behaves differently from the general-recursive operator only as applied to one not too good function. A simple example of a function γ which is general-recursive was

constructed by A. V. Kuznetsov [2]. Furthermore, it was found that an operator of this kind is far from being a rare exception: as shown by A. V. Kuznetsov, the class of functions γ, for which there exists a partially-recursive operator T_γ, general-defined in all points of a Baire space J, with the exception of a single γ, is such that is enough to substitute in $\gamma(x)$ in place of x only primitive-recursive functions x, so as to obtain all single-placed) hyper-arithmetic functions.[1] Leaning on one of the results of Kleene,[2] A. V. Kuznetsov has constructed later an example of such a partially-recursive operator, which is not general-recursive, but is general-defined in all hyper-arithmetic points.

What is required of a set of points of Baire space in order that this set could be a region of general-definiteness of any partially-recursive operator? The answer to this question is the criterion obtained (independently of each other) in November 1954 by A. V. Kuznetsov [2] and B. A Trakhtenbrot [5]. For each formulation it was found convenient to obtain a certain effectivization of the concept of open, closed and G_δ sets, on which we shall dwell in greater detail in Section 12. In terms of this effectivization, the Kuznetsov-Trakhtenbrot theorem (theorem 2) reads as follows:

For the existence of a partially-recursive operator (respectively, functional) having as the region of general-definiteness the set $M \subseteq J$, it is necessary and sufficient that M be an effective G_δ set (respectively,

1. In Kleene's generalized (by adding to the quantor prefix quantors by functions) hierarchy of forms of predicates, there corresponds to arithmetic functions also predicates (which define these functions), which are responsible in both single-function quantor forms. In the article by A. V. Kuznetsov [2] and B. A. Trakhtenbrot [5] these functions figure as "reducible to effectively closed points." In Section 8 of the present survey they are called, following A. V. Kuznetsov, B' functions.
2. The theorem that there exists a general-recursive predicate $R(\alpha, x)$ (where α is a functional variable) such that $(E\alpha)(x)R(\alpha, x)$ is true, while $(x)\,R(\alpha, x)$ is false for any hyper-arithmetic α.

an effective open set). (The proof of this theorem, proposed by V. A. Uspenskiy, was given in [13], pp. 140 -- 141).

These and other ideas and results of A. V. Kuznetsov, B. A. Trakhtenbrot, and V. A. Uspenskiy have found application to the solution of many problems, concerned with the constructivization of the concept and methods of mathematical analysis (see Section 12).

5. Two sets of natural numbers E' and E'' are called isomorphic if there exists a mapping that converts E' exactly into E'' and is realizable by a general-recursive function m = f(n), which mutually uniquely maps the natural series on itself. Important examples of non-isomorphic enumerable (but unsolvable) sets are obtained directly from the results of Post (1944).[1] At the Seminar on Recursive Arithmetic at the Moscow University (1954), A. N. Kolmogorov formulated the general problem of how rich a class of pairwise non-isomorphic enumerable (but unsolvable) sets can be.

In remarks [4], Yu. T. Medvedev answered this question, explaining how strikingly large the variety of even pairwise non-isomorphic hyper-simple sets can be. Yu. T. Medvedev obtained his answer by seeking a clear and convenient characteristic property of hyper-simple sets, which was also independently discovered by V. A. Uspenskiy [14] and A. V. Kuznetsov (ibid. p. 166).

It is natural to connect with the set M of the naturam numbers a function f, which enumerates M strictly in order of increasing of its elements. Such a function A. V. Kuznetsov proposed to call a direct enumeration of the set M. Possessing of computable direct enumerations are, as is well known (established by Post) infinite solvable, and only such sets. Of substantial interest, however, are also those cases, when the direct enumeration of this set, although not computable, is nevertheless majored by a general-recursive function. What is this class of sets for which that is impossible? And incidentally this question is given by one

1. Concerning the results of A. A. Muchnik, pertaining to isomorphic systems of enumerable sets, see Section 8.

theorem of V. A. Uspenskiy ([14], theorem 4), which we shall not give here, but with which is closely related a theorem of the characteristic property of a hyper-simple set, obtained by Yu. T. Medvedev and A. V. Kuznetsov. We give it in the formulation of A. V. Kuznetsov:

An enumerable set H is hyper-simple when and only when its complement H' is infinite but such that a direct enumeration of the set H' is not majored by any general-recursive function.

Yu. T. Medvedev introduces into consideration not the direct enumeration of the set N of natural numbers, but a different function, which he denotes E(n), which is taken to mean the number of points of the set E on the interval (1,n). If there exists such a general-recursive function θ that

$$E_2(\theta(n)) > E_1(n) \quad (n = 1, 2, \ldots),$$

then Yu. T. Medvedev says that the set E_2 is not less dense than the set E_1. This relation defines naturally the relations of uniform and large density. The fact that the set E_2 is denser than the set E_1 is written as follows: $E_1 \prec E_2$. Using this terminology, it can be said that the characteristic property of a hyper-simple set H consists of that H is an enumerable set with an infinite complement H', but a less dense one than a natural series of numbers $H' \prec N$. Since the isomorphic sets have uniform density, then the following theorem ([4], theorem 4) gives a sufficiently clear representation of the supply of non-isomorphic hyper-simple sets:

For any constructive transfinite α there exists a sequence of the type α of ever denser complements to the hyper-simple sets:

$$H'_0 \prec H'_1 \prec \ldots \prec H'_\beta \prec \ldots$$

This theorem is an obvious consequence of the circumstance noted by Yu. T. Medvedev: by adding to H or by removing from the hyper-simple merely one point, we obtain a new set (also hyper-simple) the complement to which has a different "density" than the complement to the initial sets.

6. As already noted (Section 6, item 4), the semi-lattice of degrees of unsolvability of arbitrary sets of

of natural numbers (relative to the reducibility by partially-recursive operators) have been investigated by Kleene and Post. This still did not solve Post's problem, since the latter pertains especially to enumerable sets. The complexity of constructing a system of enumerable sets should bring to mind, however, the existence of different degrees of unsolvability also in the system of enumerable sets. We have already remarked (Section 6, item 4) that Yu. T. Medvedev formulated the problem of reducibility as a mass problem in his definition. But the mass problem in the sense of Yu. T. Medvedev is the problem of the construction of an arithmetic function, satisfying definite requirements. Of what function can one speak in Post's problem of reducibility? The Post problem can be formulated as follows: it is necessary to ascertain whether for any pair of enumerable but not solvable sets A and B there exists a partially-recursive operator T which transforms the set A (its characteristic f_A) into a set B (the function f_B). But a partially-recursive operator is still in itself not an arithmetic function. It is, however, difficult to give a functional representation for a partially-recursive operator. Such representations were proposed by various authors. The best known is that proposed by Kleene.[1] Other representations are contained in the dissertation of Yu. T. Medvedev,[2] in the work by A. V. Kuznetsov [2] and B. A. Trakhtenbrot [5], and in the article by V. A. Uspenskiy [13], who identify computable operators with constructively continuous functions on a generalized Baire space (concerning this, see Section 10). Yu. T. Medvedev considers operators which convert everywhere-defined arithmetic functions (i.e., sequences of points of ordinary Baire space) in everywhere-defined arithmetic functions. By means of the conversion, realizable by operator T, which converts the function f into the function g, there is induced a conversion of each cortege f (0), f(1,...,f(n) into a certain cortege g(0), g(1),...,g(m), and

1. See S. C. Kleene. Introduction into Meta-Mathematics, Moscow, Foreign Literature Press, 1957, p. 308.
2. Yu. T. Medvedev, Degrees of Difficulty of Mass Problems, Dissertation, Moscow State University, 1955.

vice versa, such a conversion of corteges $a_1, a_2, \ldots, a_n$ into corteges $b_1, b_2, \ldots, b_n$ at which there occurs with increasing n, at the very most, only an addition to an already obtained cortege of new terms, defines also a conversion of one function (infinite sequences) α into another one β. Corresponding to the operator T, which converts functions into functions, is thus (mutually uniquely) an operator Δ, which converts corteges into corteges. But all pairs of corteges can be effectively renumbered, and thereby even with a primitive-recursive function. Thus there will correspond to the set of all pairs of corteges, defined by the operator Δ (and consequently, to the corresponding operator T) a certain sequence of numbers, i.e., an everywhere defined arithmetic function ∂, about which we shall, together with Yu. T. Medvedev and A. A. Muchnik, that it "realizes the functional representation of the operator T."

As proved by Yu. T. Medvedev,[1] for each partially-recursive operator T there exists a primitive-recursive function δ, which realizes its functional representation.

Furthermore, for all partially-recursive operators T, which convert the predicates (i.e., the characteristic functions of sets) into predicates, there exists a universal functional representation, i.e., a primitive-recursive function $\varphi(x, w)$ such that the functional representation of the operator T is realized by the function $\varphi_x(w) = \varphi(x, w)$. The operator corresponding to this function will be denoted T_x.

To each pair $\{A, B\}$ of sets there corresponds a class of functions φx of operators T_x, which convert any one of the characteristic functions f_A or f_B into another. Since the functions φ_x are primitive-recursive (i.e., they are known to be general-recursive), then the (mass) problem of reducibility of at least one of the sets A, B into another is not solvable when and only when the class of functions φx or of operators T_x corresponding to it is empty, i.e., when not one of the sets A, B reduces to another by partially-recursive operators. An example of

1. Yu. T. Medvedev, Degrees of Difficulty of Mass Problems, Dissertation, Moscow State University, 1955.

such two enumerable but not solvable sets H_1 and H_2, for which the corresponding class T_x was empty, was indeed constructed by A. A. Muchnik ([1],[1] theorem 1) who solved thereby the Post problem. The results of A. A. Muchnik prove the existence of incomparable with each other degrees of unsolvability of enumerable sets. Furtherfore, the sets constructed by A. A. Muchnik are hyper-simple. Thus, even in the system of hyper-simple sets there exist sets with incomparable degrees of unsolvability. A. A. Muchnik has strengthened considerably this result, by proving ([1], theorem 2) that there exists an enumerable sequence of hyper-simple $H_1, H_2, \ldots, H_n$, the terms of which are pairwise not reducible to each other by partially-recursive operators. At even further strengthening it the rather unexpected[2] (inasmuch as both in the structure of degrees of difficulties of Yu. T. Medvedev (see Section 6, item 4) and in the semi-lattice of degrees of unsolvability of Kleene-Post, there exist minimal (non-vanishing) degrees of unsolvability[3]), is the theorem 3, proved by A. A. Muchnik, that no what the enumerable but unsolvable set G may be, there exists a hyper-simple H, of smaller degree of unsolvability, i.e., one that reduces to G but to which G does not reduce by a partially-recursive operator. As noted by Harley Rogers,[4] this result is missing from the work of the American mathematician Friedberg, who also solved (independently of A. A. Muchnik) the Post problem. But indeed this result is of particular interest in connection with the Post problem,

---

1. The heading of the note ("Unsolvability of the Problem of Reducibility of the Theory of Algorithms") is explained by the statement of the problem clarified above.
2. See, for example, Abstract by Hartley Rogers (in the Journal of Symbolic Logic, 22, (1957), pp. 218 -- 219).
3. In the lattice of degrees of difficulties, this is the degree of difficulty of the construction of a function which is not general-recursive. With respect to the Kleene-Post semi-lattice see: Clifford Spector, On Degrees of Recursive Unsolvability, Journal of Symbolic Logic, 21 (1956), p. 111.
4. See the abstract by H. Rogers, Journal of Symbolic Logic, 22, (1957), p. 219.

since it follows from it that among the unsolvable enumerable sets there is none that can serve as a standard, suitable for proving the unsolvability of any unsolvable enumerable set.

8. Descriptive Properties of Arithmetic Sets. Problems of Classification of Sets, Functions, and Other Objects.[1]

1. P. S. Novikov and his students spend much time on questions connected with the analogy between the enumerable sets of natural numbers and A sets. The solvable sets in this analogy correspond to B sets. This analogy is based on the fact that enumerable sets are obtained from solvable by projection, i.e., the same way that A sets are obtained from B sets. It is natural that many questions and concepts arose, analogous to those studied in the descriptive theory of sets. An example of such a concept is the universal enumerable set, an example of which is the question of separability of enumerable sets by solvable ones. But while the universal enumerable set is perfectly analogous (in its definition and role) to the universal A sets, from the answer to the question of the separability there is already obvious a certain violation of the complete analogy. Indeed, P. S. Novikov, and later B. A. Trakhtenbrot [2] constructed many examples of pair of non-intersecting enumerable sets, which are recursive-non-separable (i.e., not separable by solvable sets[2]).

While any two non-intersecting A sets are separable by B sets, it is not the analogies of A sets, i.e., enumerable sets, that are recursive-separable but their complements, i.e., the analogues of CA sets. Connected with the fact of separability for the latter was indeed the first example of a pair of non-intersecting recursive-non-

1. This section was written with the collaboration of A. V. Kuznetsov.

2. The first example of this kind was published by Kleene in 1951, who obtained it independently of P. S. Novikov.

separable enumerable sets, constructed by P. S. Novikov[1] in analogy with his example of CA sets, not separable by B sets.

At the same time it was possible to solve for enumerable sets several problems, the analogies of which in the theory of A sets entail principal difficulties. One of such problems, formulated by P. S. Novikov, was solved by A. A. Muchnik [2]. We proceed to a discussion of this problem now.

2. In the theory of enumerable sets one frequently makes use of the following modification of the ordinary concept of the universal enumerable set: an enumerable set U of natural numbers is called universal, if for any enumerable set V of natural numbers there exists such a mutually-unique general-recursive function φ, that

$$\forall x (x \in \equiv \varphi(x) \in U).$$

In all the above-mentioned examples of recursively inseparable enumerable sets, the latter ones are universal. P. S. Novikov raised the question of whether there exist non-universal recursive non-separable enumerable sets. An answer to this question was obtained by A. A. Muchnik [2], who introduced a concept of strongly non-separable enumerable sets and who proved the existence of such. Indeed, it was found that strongly non-separable sets are recursively non-separable, but not universal. With the aid of these A. A. Muchnik obtained, in addition, a positive answer to two problems raised by V. A. Uspenskiy ([5], problems I and II), which we shall discuss in connection with problems of incompleteness and incompleteability of logical calculus (see Section 12).

In the same paper [2] A. A. Muchnik introduced into consideration such sets ("sets of pairs") for which there exist sets that are strongly non-separable from them, and proved many theorems concerning these and also concerning the universal and simple sets of Post. He also observed still another example of violation of the analogy between A sets and enumerable sets ([2], theorem 10).

To one of the questions raised by A. A. Muchnik in

1. See the editor's remak on p. 277 in the book of Kleene (Meta-Mathematics).

the same place ([2], Section III, problem 2), he later on obtained an answer, which states that any unsolvable enumerable set can be represented in the form of a union of two recursively non-separable sets. Since such a representation is impossible for solvable sets, this establishes a direct connection between non-solvability and non-separability.

Many other results were obtained by A. A. Muchnik pertaining to properties of enumerable sets and are contained in [1, 2].[1]

3. The idea of the analogy between enumerable sets and A sets arose in connection with the work by Kleene, in which a certain classification ("hierarchy") was constructed for "elementary"[2] predicates by their quantor prefixes. As noted in the work by Kleene, the analogy between logical operations, expressed by Quantors of existence and generality, and the geometrical operations of projection and intersection (respectively) is known. (He had in mind here the works by Tarski and others). Leaning on Goedel and Ulem, Kleene advanced the suggestion of the possibility in connection between his results and the theory of Borel and Baire. In a survey paper to the Moscow Mathematical Society on 4 December 1945 ("Computable Sequences and ther Value in Investigations of the Fundamentals of Mathematics") A. N. Kolmogorov has concretized this analogy, relating the results of Kleene concerning the existence of an enumerable but unsolvable set, with the well-known theory of M. Ya. Suslin, on the existence of an A set which is not a B set. In the winter 1946/1947 of the academic year, a seminar was in session under the leadership of P. S. Novikov, in which problems were investigated connected with this analogy.

---

1. Certain of these have been published together with detailed proofs in the Seventh volume of Trudy Moskovskovo Matematicheskovo Obshchestva (Works of the Moscow Mathematical Society).
2. In the Russian translation of the book by Kleene these predicates are also called "arithmetic in the sense of Goedel." In this section they (and also the sets that they define) will be called briefly "arithmetic." In many papers of Soviet authors they were called recursive-projective.

P. S. Novikov was interested in these questions in connection with the difficult problems in descriptive theory of sets, which we discussed in Section 1 of the present article. The greater simplicity of the nature of subsets of the natural series compared with the subsets of the continuum has given rise to the hope that the solution of problems in the theory of "arithmetic" sets will suggest the answers that can be expected for analogous questions in descriptive theory of sets. In particular, the hypothesis of the existence of a non-denumerable CA set without a perfect kernel (see Section 1, item 1 of the present article) was considered as a corresponding Post theorem on the existence of a simple set (enumerable set, the complement to which is infinite, but does not contain an enumerable subset) (see Section 7, item 2).

The carrying out of an analogy between the aforementioned classification of Kleene (it was known to us for a long time only from the work of Kleene, although actually it is the Kleene-Mostowski hierarchy) and the classification of projective sets has made it possible to raise, in the theory of "arithmetic" sets, questions analogous to the problem of separability of projective sets of higher classes. A solution of these problems involves no difficulty. The loss of separability for sets, defined by predicates with n-quantor prefixes of the type $\exists x_1 \forall x_2 \exists x_3 \forall x_4 \ldots$ and a recursive subquantor portion R (A. Mostowski denotes subsets by the symbol P_n and their complements by the symbol Q_n) were found to be (for n = 1, 2, 3,...) analogous to the laws of separability for A sets, established earlier by P. S. Novikov, but not analogous (this was already noted in above in item 1) to the laws of separability for A_1 sets (i.e., A sets). In spite of the latter, the analogy (in the classifications) between P_n sets and A_n sets predicted the hypothesis that for $n \geqslant 2$ the laws of separability for A_n sets are analogous to the laws of separability for A_2^n sets, the non-contradiction of which was indeed proved later by P. S. Novikov, for sufficiently large n (see Section 1, item 1).

Incidentally, many other questions in the theory of arithmetic sets, analogous to the well-known problems of descriptive theory of sets, were raised and solved

along with the above. Many results (including those mentioned above) were expounded in detail in the lectures by P. S. Novikov, which he delivered at the Moscow University in the spring of 1952. Among these are included also the results pertaining to problems in the laws of unformization for certain classes of "arithmetic" sets. In particular, the question was raised of the possibility of separating in any set of a given class a uniformizing subset of the same class. P. S. Novikov has shown in his lectures that for classes of primitive-recursive, solvable, and enumerable sets this question is answered in the affirmative, wherein the case of the first two of these classes the uniformizing subset of a given set can be taken to be the set of his lower points. Later on V. A. Uspenskiy [14] proved that in the case of a class of enumerable (i.e., P_1) sets the latter, generally speaking is not true, since there exists such an enumerable set, the set of lower points of which is not enumerable.

4. The already noted incompleteness of the analogy between the enumerable sets and A sets as a part of the general analogy between the P_n and A_n sets has raised questions of the refinement of the limits of the analogy for the purpose of its subsequent perfection or replacement by another, deeper analogy. Thus, P. S. Novikov expressed the idea that the analogy is incomplete only for the beginnings of the classifications as a consequence of the specific properties of A sets compared with the properties of projective sets of higher classes. On the other hand, A. V. Kuznetsov even in January 1949 (Seminar on Mathematical Logic at the Moscow University) attempted to lay the grounds for an analogy of a different kind, at which the class of A sets is compared not with the class of enumerable sets, but another broader class. The latter consisted of the results of the application to solvable sets of an operation analogous to the well-known A operation of P. S. Aleksandrov, and, as was proved later, was broader than the class of "arithmetic" sets. However, the fact that other aspects of this analogy have not been worked out has hindered the possibility of making it sufficiently convincing. In addition, at that time it was apart from the principal trend in the work on

mathematical logic, which had a greater tendency of concentrating around the problem of the theory of algorithms and constructivization of mathematics. The value of the development of problems connected with classification of sets which are not "arithmetic" to provide answers for many questions in the constructive mathematics itself, begin to become clear only later.

Nevertheless, A. V. Kuznetsov refined gradually the principles of the new analogy, proposed by him. He first noted that it is possible to carry out a sufficiently good analogy between Kleene's hierarch of "arithmetic sets" and the Borel hierarchy of those classes of B sets, the numbers of which are finite in the latter hierarchy. This analogy was based on the correspondence between the quantors of generality and existence (relative to numerical variables) and the operations of (denumerable) intersection and joining. In addition, he established thereby a closely related analogy between Baire's hierarchy of the first ω classes of B functions and a certain natural hierarchy of "arithmetic" functions, i.e., functions the graphs of which are "arithmetic" sets). The latter are classified here (as in the case of a Baire hierarchy) by the number of symbols of the limiting transitions, with, as shown during the same time (1949 -- 1951), the n-th class includes those and only those "arithmetic" functions, the graphs of which are simultaneously also P_{n+1} and Q_{n+1} sets. Later on, (already in 1952) A. V. Kuznetsov simplified the above-mentioned analogue of the A operation (after which it appears as follows

$$S(x_1, \ldots, x_k) \equiv \exists\varphi\forall y R(\varphi(y), \varphi(y'), x_1, \ldots, x_k),$$

where S is the result of applying an operation to the predicate R) and constructed a corresponding analogue between the Luzin sieve operation, and also showed that with the aid of the existence quantor over a variable that runs over a continual region (for example, the region of numerical predicates or numerical functions), it is possible to construct an analogue of the ordinary projection operation, which serves as the basis for the Luzin classification of projective sets. All these operations (the analogues of A operation, net and projection) were found to yield upon

single application to general-recursive predicates the same class of predicates, which A. V. Kuznetsov called A' predicates. The laws of separability for (the corresponding) A' sets were found to be quite analogous to the laws of separability for A sets. With this, the analogues of B sets, called B' sets, were defined analogously to the ordinary manner (i.e., as sets which are simultaneously both A' sets and complements of such sets). For the sieve analogue there was constructed an apparatus of indices, analogous to the well-known one for the ordinary sieve, and it was noted that upon application of the sieve to solvable sets (to general-recursive predicates) the indices cannot be different from primitive-recursive (in that sense, that they are transfinite, defined by primitive-recursive orderings). From this A. V. Kuznetsov obtained in the same year, 1952, the following theorem:

Any B' transfinite is primitive-recursive. (A trivial consequence of this is that any A' transfinite is primitive recursive.)

It was thus found above all that the general concept of a constructive transfinite cannot be described even by B' predicates: in the opposite case one could by the diagonal procedure construct a B' transfinite which is not constructive (i.e., which is known to be not primitive-recursive). This example is a good illustration of the great role that can be played by considerations, which appear to be utterly non-constructive, for the clarification of the nature of constructive mathematical objects and theory.

Later on (1955) A. V. Kuznetsov observed that the concept of the B' function is closely related with the regions of general-definiteness of partially-recursive operators (see Section 7 of the present article). In the joint work of A. V. Kuznetsov [2] and B. A. Trakhtenbrot [5] there was introduced the concept of an effectively closed point of Baire space J. It was found that the effectively closed point is such a point for which there exists a partially recursive functional, general-defined everywhere in J, in addition to this point, and that B'

functions[1] are such functions, which are primitively-recursive relative to the functions that are effectively closed points in J. From this it follows that no matter how far in the classification of B' functions, for example in the transfinite classification of Kleene-Mostowski, there are effectively closed points and this means that there exist partially-recursive functionals, which are generally defined everywhere with the exception of these effectively closed points, which as far as desired, i.e., partially-recursive functionals, which are as close as desired to general-recursive. This again cannot be observed by purely constructive means, although we are speaking of objects that have a direct relation to the theory of algorithms.

Somewhat later (1952 -- 1956), A. V. Kuznetsov noted that it is possible to construct the arithemetic analogies of C sets, investigated by Ye. A. Selivanovskiy -- C' sets, obtained from solvable ones by alternating the operations of sieve and complementation (respectively A' operations and complementation), the class of which was found to be narrower than the class of analogues of projective sets (i.e., sets which Kleene called later on "analytic"). Indeed, the analogue of the well known theorem that any C sets belong already to a second class of projective sets, was found to be true.

5. By virtue of the well-known Goedel theorem, any "arithmetic" predicate can be obtained with the aid of operations of narrow calculus of predicates from the predicates "x + y = z+ and "x · y = z." It is natural to raise the question of what predicates and what means can yield all the B' predicates, and their analogues in the descriptive set theory -- B predicate -- and also the analogues of "arithmetic" predicates in the sense of the previous analogy -- projective predicates. Examples of answers to these questions (or on the corresponding questions concerning the functions) are the following theorems obtained by A. V. Kuznetsov.[2]

1. In the work of A. V. Kuznetsov [2] and B. A. Trakhtenbrot [5] the B' functions are referred to as "functions reducible to effectively-closed points."
2. Certain of these, and also some earlier mentioned results were reported later on at the Third All-Union Mathematical Congress (July 1956).

I. From the predicates "$x + y = z$," "$x \cdot y = z$" (considered as those following in the object region of real numbers), "x -- integer" and all possible predicates of the form "$x = c$" where c is a real number, it is possible to obtain, with the aid of the operations of the narrow calculus of predicates, all the projective predicates and only these predicates.

II. For any B' predicate S there exists such a B' predicate T and such a formula of narrow calculus of predicates $\mathfrak{A}(P_1, P_2, P_3)$ which after insertion in the place of the predicate variable P_3 of the predicate "$x' = y$"[1] uniquely defines (on the object region of the natural numbers) a pair of predicates S and T, i.e., it becomes equivalent to the statement "P_1 is S and P_2 is T."[2]

III. For any B' function f there exists such a B' function g and such a system of functional equations $\mathfrak{S}$, which uniquely defines on the region of natural numbers a pair (of everywhere defined) functions f and g and, apart from the signs of these defined functions and the functions x', contains no other signs of functions.

IV. The same as in III, with replacement of B for B' everywhere, the natural numbers of real numbers, and the functions x' by functions $x + y$, $x \cdot y$, and all possible constants.

A. V. Kuznetsov raised questions concerning the continuation of the foregoing analogies further and concerning search for new analogies. In particular, the question was raised of what is the class of those predicates, defined on the object region of real numbers, which can be obtained from projective predicates (or from the much more simpler ones, listed in theorem I of the present item), with the aid of operations of calculation of predicates of the second degree (i.e., with quantors by variable predicates). As noted by A. V.

---

1. The function "prime" relates to the natural number x the following number $x + 1$.
2. See N. N. Luzin, Certain New Results of Descriptor Theory of Functions, Moscow-Leningrad, ONTI, 1935.

Kuznetsov (1952), the class of such predicates is broader than the class of predicates corresponding to sets that enter in the well-known classification of effective sets by P. S. Novikov.[1]

6. In studying various types of definitions of the concepts of the B' functions, A. V. Kuznetsov observed that the B' function can be characterized in a certain sense as computable with the rule of infinite induction.[2] In fact: for any B' function $f(x_1, \ldots, x_n)$ there exist such g and $\mathfrak{S}$, which satisfy all the conditions of the theorem III, item 5, and in addition, the following condition: by means of rules a) of substitution of terms, b) replacement of a term by an equal term, and c) rule of infinite induction, for all natural numbers $a_1, \ldots, a_n$, b the equality $f(a_1, \ldots, a_n) = b$ is derivable from $\mathfrak{S}$ when and only when it is true. This was generalized by A. V. Kuznetsov at the Third All-Union Mathematical Congress in July 1956. On the same day and in connection with these results, and also several results reported at that time by B. Ya. Falevich (see Section 12, item 8) at the Session of the Section of Mathematical Logic of the Congress, a conversation arose on the possibility of constructization of these results. P. S. Novikov proposed a constructivized variant of the rule of infinite induction, which was called the "constructive Carnap rule," or rule of constructively-infinite induction. P. S. Novikov, raised, in particular, the question of whether it is enough to add this rule to the ordinary logical-arithmetic calculus, in order to make the latter one complete. This question was answered in the affirmative in the fall of the same year by A. V. Kuznetsov (reported in detail at the Seminar on Mathematical Logic at the Moscow University, February-March 1957 and at the Session of the Moscow Mathematical Society on 12 March 1957 [5]). He arrived

---

1. See N. N. Luzin, Certain New Results of Descriptor Theory of Functions, Moscow-Leningrad, ONTI, 1935.
2. The rule of infinite induction (in other words, the Carnap rule) consists of the following: if all the formulas of the sequence $\mathfrak{A}(0), \mathfrak{A}(1), \ldots, \mathfrak{A}(n), \ldots,$

are proved, then the formula $\forall x\, \mathfrak{A}(x)$ is also proved.

at this result by descriptive estimates of the corresponding classes of formulas, detailed analysis of the analogue of the sieve operation, and the chain of reducing certain calculus to others, although in the final proof the first two of these items were eliminated.

The corresponding completeness theorem was proved thereby also for the case when the logical-arithmetic calculus contains such formulas with predicate and functional variables, although in the latter case only on the one side: any identically true formula is provable. For the refutability[1] of any formula in the last calculus, which is not identically true, the addition of the above-mentioned rule was found to be insufficient, owing to the fact that the set of such formulas is not a CA' set (i.e., is not a complement of the A' set).

The question of the constructiveness of the formulation of the result of A. V. Kuznetsov (at least for the cases of ordinary logical arithmetic calculus) has given rise to a discussion (at the Seminar on Mathematical Logic), arising at the initiative of A. A. Markov, which casts doubts on the constructiveness of the definition of the rule of constructively-infinite induction in view of its connection with the general concept of constructive transfinite, the definition of which, as noted above (item 4), contains non-constructive moments. In this connection P. S. Novikov, in a paper read at the Seminar on Mathematical Logic on 22 May 1957, indicated several ways for further constructivization of the concept of constructive transfinite, the concept of a formula provable with the rule of constructively-infinite induction, and an entire class of related concepts.

1. The refutability of a formula is taken here to mean, for example, in that sense, that its addition as a new axiom makes the calculus contradictory (it is assumed that the rule of substitution is contained in it).

Chapter III

MATHEMATICAL APPLICATIONS OF THE THEORY OF ALGORITHMS

9. Algorithmic Questions of Algebra[1].

1. As soon as an exact definition of the concept of the algorithm was found, it became possible to prove the presence of unsolvable algorithmic problems in mathematics. This was first made by the American mathematician Church, who proved the unsolvability of the "general problem of solvability" for the calculus of predicates. The algorithmic problem the unsolvability of which was proved by Church, is formulated in mathematical logic itself. This result alone could not solve the problem of the place of the unsolvable algorithmic problems in mathematics. The following question remained unclear: can such problems be actual algorithmic problems, concerning very widely spread functional concepts of mathematics?

This question was solved partially in 1947 by the Soviet mathematician A. A. Markov [30] and by the American E. Post, who simultaneously and independently of each other constructed examples of associative systems, i.e., semigroup) with an unsolvable problem of identity. The problem of identity is connected with one of the widespread methods of specifying algebraic systems (groups, semigroups, etc.), namely specifying them with the aid of forming and defining relations. With this, the elements of the specified system are represented by words made up of formants, while the same element is represented by an entire class of words which are generally speaking different in their notation. The words that represent the same element in such a specification, are called "equivalent" or "equal." The problem of identity, which is formulated for finite-definite systems, i.e., for systems, specified by a finite number of formants and relations, consists of finding an algorithm that makes it possible to ascertain whether an arbitrary

1. This section was written by S. I. Adyan.

pair of words of the considered system are equivalent to each other in this system or not.

By unsolvability of any particular algorithmic problem is understood the impossibility of a corresponding algorithm.

The proof of the unsolvability of this problem of identity for associative systems made it possible to solve the question of solvability of many other algorithmic problems, concerning groups, semigroups, matrices, etc. An example of a semigroup with construction with an unsolvable identity problem was constructed by the Englishman Turing in 1950.

It should be noted that the concepts of an associative system (semigroup) and a semigroup with contraction appeared in mathematics as a result of a logical analysis of the concept of a group, which is one of the most important and widely spread oncepts in mathematics. Therefore the problem of identity in group theory, formulated already in 1912, has occupied a place of special significance among other problems of this kind. It was studied by many Soviet and foreign mathematicians. In 1952 the Soviet mathematician P. S. Novikov [27] proved the unsolvability of this problem, by constructing an example of a finite-definite group with unsolvable problem of identity. In 1957, a paper by P. S. Novikov [30], devoted to the proof of this result, was awarded the Lenin prize: as expected, on the basis of this result within a relatively short time, numerous other results were obtained by P. S. Novikov, A. A. Markov, S. I. Adyan, G. S. Tseytin, and K. A. Mikhaylova. These results will be discussed later.

This has led to the formulation of a new trend in mathematics, engaged in problems of existence of certain algorithms.

2. Algorithmic problems for associative systems were studied by A. A. Markov and his student G. S. Tseytin.

The problem of right (left) divisibility for associative systems is formulated in the following manner: it is required to find an algorithm, by means of which one

could for any two elements Q and R of the system $\mathfrak{A}$ recognize whether there exists in $\mathfrak{A}$ such an element X, that the relation XQ = R (or respectively QX = R) is satisfied in the system $\mathfrak{A}$. An example of an associative system with unsolvable problem of right divisibility was constructed by A. A. Markov [32] in 1947, and in this associative system were solved both the problem of identity and the problem of the left divisibility.

A. S. Markov in 1951 [44] proved the unsolvability of the general problem of commutability of matrices, which is formulated in the following manner: let $U_1, U_2, \ldots, U_q$ be square integer matrices of order n. We shall say of a matrix U of order n that it can be represented in terms of $U_1, U_2, \ldots, U_q$ if there exist natural numbers $\lambda, i_1, i_2, \ldots i_\lambda (1 \leqslant i_\nu \leqslant q)$ such that

$$U = \prod_{\nu=1}^{\lambda} U_{i_\nu}.$$

The general problem of representability for matrices of order n consists of finding an algorithm, by means of which it is possible to recognize, for any system of matrices $U_1, U_2, \ldots, U_q$ of order n, whether an arbitrary matrix U is representable in terms $U_1, U_2, \ldots, U_q$.

The particular problem of commutability for a fixed system of matrices $U_1, U_2, \ldots, U_q$ of order n consists of finding an algorithm, by means of which it would be possible to reconize whether any matrix U of the same order is representable in terms of $U_1, U_2, \ldots, U_q$. In 1951 A. A. Markov [44] constructed a system of 102 matrices of sixth order, for which the particular problem of representability is unsolvable, i.e., the algorithm sought in the problem is impossible. It followed from this that the general problem of representability is unsolvable for any $n \geqslant 6$. Later on (at the Seminar on Mathematical Logic at the Moscow University, October 1957) A. A. Markov showed that the general problem of representability is unsolvable already for $n \geqslant 4$.

The simplest example of an associative system with unsolvable problem of identity, given by A. A. Markov [48], contained 13 forming and 33 defining relations. G. S. Tseytin in 1956 [3], leaning on the result of

P. S. Novikov concerning the existence of a group with unsolvable problem of identity, constructed an associative system with five forming elements and seven defining relations and with an unsolvable identity problem. In recent time, using this example of G. S. Tseytin, A. A. Markov constructed a system of 27 matrices of sixth order, for which the corresponding particular problem of representability is unsolvable. The same result by G. S. Tseytin makes it possible to simplify the example of the group of P. S. Novikov with unsolvable problem of identity.

By K-system we shall take to mean henceforth a finite-definite associative system, i.e., associative systems with finite alphabet, and a finite number of defining relations.

An arbitrary property $\mathfrak{a}$ of K systems is called invariant if any K system, which is isomorphic to any other K system with a property $\mathfrak{a}$, has in itself that property. For any invariant property $\mathfrak{a}$ and any finite alphabet A, there is formulated the problem of recognition of the property of $\mathfrak{a}$ for the alphabet A. It is required to find an algorithm by means of which one could for any finite system of defining relations, written in the alphabet A, indicate whether the K system defined by it has the property $\mathfrak{a}$. In 1951 A. A. Markov [43] proved the following theorem.

Let $\mathfrak{a}$ be an invariant property of K systems. If there exist both a K system with this property as well as a K system that is not included in any K system with this property, then there exists an alphabet, for which the problem of recognizing the property $\mathfrak{a}$ is unsolvable. If at the same time there is a K system with property , defined in p letters, then for alphabet with number of letters greater than p + 4 the problem of recognizing the property is unsolvable. Among the properties $\mathfrak{a}$ which satisfy the conditions of this theorem, are, for example, the following properties: 1) unitarity, 2) finiteness, 3, semigroup property, 4) inclusion in group calculus, 5) solvability of the identity problem, etc.

A general theory of associative calculi, which represent the specification of K-system with the aid of forming and defining relations, together with a proof of his foregoing results, were treated by A. A. Markov in

his monograph [48].

G. S. Tseytin [2] improved the last result of A. A. Markov, proving that the theorem is true for alphabets which contain not less than p + 2 letters. He proved in the same place that this theorem in the formulation given cannot be extended to a p + 1 letter alphabet, it proved that for any $p \geqslant 0$ there exists an invariant property U_p of K systems such that there exists a K system not included in any K system with the property U_p, and there exists a K system in a p-letter alphabet, having the property U_p, but for any alphabet containing more than p + 1, the problem of recognition of the property U_p is solvable. Such a property U_p is the following property: "To be isomorphic to a free associative system with p formants."

From 1954 through October 1957 a seminar was in session at the Moscow State Pedagogical Institute imeni V. I. Lenin on algorithmic problems of algebra, under the leadership of P. S. Novikov (henceforth identified as the "Seminar of the MGPI"). Many of the results on algorithmic problems, concerning groups, semigroups, universal algebras, etc., were obtained by members of this seminar P. S. Novikov, S. I. Adyan, A. V. Kuznetsov, and K. A. Mikhaylova.

To prove the unsolvability of the problem of identity of group theory, P. S. Novikov developed a theory of transition and quadratic-transition letters, proving a system of lemmas on the transformation of words into groups with participation of the indicated letters. These lemmas, which permit establishing the inequality of various types of letters in groups, have found application in the proof of other algorithmic and purely-algebraic results of P. S. Novikov and S. I. Adyan.

A trivial consequence of the unsolvability of the identity problem of group theory, proved by P. S. Novikov [30] is the unsolvability of the well-known general problem of conjugateness. A particular problem of conjugateness for a given group F consists of finding an algorithm, which permits for any two elements X and Y of group F to recognize whether they are conjugate to each other in F or not (two elements X and Y of group F are called conjugate in F if and only if there exists in F such a third element

Z such that $X = ZYZ^{-1}$ in group F). In the general problem of conjugateness it is necessary to find a single algorithm for all groups. Here the unsolvability of the general problen is also proved by indicating an example of an unsolvable particular problem, namely, in a group with an unsolvable problem of identity one cannot solve the problem of conjugateness. P. S. Novikov [29] has also constructed a simple example of a group with unsolvable problem of conjugateness.

Both the problem of identity and the problem of conjugateness of group theory have a topological interpretation. As is known, any finite-definite group is a fundamental group of a certain polyhedron. To each word in the fundamental group there corresponds a closed path on the polyhedron, passing through a fixed point O. With this, two closed paths, passing through a point O, will be connectively homotopic (i.e., continuously deformable one into another for a moving point O) if and only if the corresponding words of the fundamental group are conjugate to each other. From this we obtain the following interpretation of the results of P. S. Novikov.

There exists such a polyhedron that it is impossible to obtain an algorithm that would permit for any pair of paths, passing through a fixed point of the polyhedron, to recognize whether they are connectively homotopic to each other or not (respectively for problems of conjugateness: whether they are freely homotopic to each other or not).

4. The general problem of isomorphism consists of finding an algorithm, which would permit for any pair of groups, specified with the aid of a finite number of forming and defining relations, to recognize whether they are isomorphic to each other or not.

A particular problem of isomorphism, formulated for a specified finite-definite group F, consists of finding an algorithm that would permit for any finite-definite group to recognize it is isomorphic to the group F or not.

P. S. Novikov, on the basis of his work [30] proved (March 1955), (Seminar of MGPI) the unsolvability of the general problem of isomorphism. His student S. I. Adyan

[1], leaning on the same investigation, proved in 1955 that the particular problem of isomorphism for any finite-definite group is unsolvable. At the same time he proved the unsolvability of a broad class of algorithmic problems of recognition of group properties. In particular he proved the non-recognizability of such group properties as unitarity, finiteness, periodicity, commutativity, simplicity, nilpotence, solvability, freedom, ability of having a free subgroup, etc. In the proof of these results S. I. Adyan introduced the concept of a quasi-transition letter and proved for it several lemmas, analagous to the lemmas of P. S. Novikov, for transition letters, and particularly the lemma of the exclusion of the insertions of a quasi-transition letter in sequences of conversions of words. Unlike the transition letters, the quasi-transition letters have that characteristic feature, that even as a result of transformations that do not contain insertions of these letters, their number can increase without limits. They so to speak "generate each other."

Leaning on one of these results (specifically, on the non-recognizability of unitarity groups), A. A. Markov proved the unsolvability of the problem of homeomorphism of polyhedra. He constructed such a four-dimensional polyhedron, that it is impossible to obtain an algorithm which would define for any arbitrary four-dimensional polyhedron whether it is homeomorphic to this polyhedron or not. (The polyhedron constructed by him is a maniforld). This result was reported by A. A. Markov at the Seminar on the Applications of Theory of Algorithims at the Mathematical Institute imeni A. A. Steklov on 23 January 1958.

Using the same methods as in reference [1], S. I. Adyan [5] proved the following theorem on unsolvable problems of recognition of group properties, analogous to the above-mentioned theorem of A. A. Markov on unsolvable problems of recognition of properties of associative systems:

Let α be a certain invariant group property. If there exists both a finite-definite group having the property α and a finite-indefinite group which cannot be imbedded in any finite-definite group with this property, then it is impossible to obtain an algorithm that would permit us to recognize whether any finite-definite group F

has the property ✱ or not.

An algorithm that defines, for any finite-definite group, whether it coincides with its commutant (i.e., whether it becomes unit group or if one adds the commutation relation for the formants), is very simple to construct. This is an algorithm that permits to ascertain whether any Abel group is unitary or not. Therefore from the last theorem of Adyan it follows that any finite-definite group is imbeddable in a finite-definite group, which coincides with its commutant.

The class of finite-definite groups we shall call complete if any finite-definite group is isomorphic with any other subgroup of a certain group of this class. It is obvious that in every complete class of groups there is a group with an unsolvable identity problem. Consequently, for any complete class of groups it is impossible to obtain an algorithm that solves the problem of identity for all groups of this class. S. I. Adyan calls a class of finite-definite groups K as effectively complete, if there exists an algorithm, which in accordance with any finite-definite group F indicates a group F* from the class K, with a certain subgroup to which F is isomorphic and furthermore in such a way, that if the group F belongs to class K, then the corresponding group F* coincides with F. He established that the theorem proved in reference [5], on the unsolvability of problems of recognition of group properties, remains in force if instead of the class of all finite-definite groups (which obviously is effectively complete) one considers any prescribed effectively-complete class of groups, i.e., if one requires of the sought algorithm that it solve the problem not for all finite-definite groups, but also for groups of a specified effectively-complete class (Seminar on Application of the Theory of Algorithms at the Mathematics Institute imeni V. A. Steklov, 20 February 1958).

We shall call a finite-definite group as conditionally-unitary relative to a given system of identical relations, if the addition to the relation of this group of the considered identity relations converts it into a unit group. S. I. Adyan proved that the class of groups, which are conditionally unitary relative to any given system of non-trivial identical relations (i.e., relations

which are satisfied not in any group), is complete.

Proved in the same place that the class of finite-definite groups, specified by a finite system of mutually-conjugate formants, is complete. Both theorems are proved by effective embedment of an arbitrary finite-definite group in a group of a class, the completeness of which is proved.

5. In 1955, S. I. Adyan [2] proved the non-solvability of the problem of right (left) divisibility for semigroups with contraction (as indicated above, the corresponding problem for associative systems was solved by A. A. Markov). The problem of whether there exists in a half group with contraction and non-solvable problem of right (left) divisibility and solvable problem of identity, remained open, since in the semigroup given by S. I. Adyan with contraction, the problem of identity is also not solvable.

The particular problem of the existence of an inverse element for a given semigroup G with rule of cancellation and with unity lies in discovering an algorithm, which would permit for any element $X \in G$ to indicate whether there exists in G an element inverse to it, or not (the element $Y \in G$ is called inverse for $X \in G$ if and only if the equality XY = 1 is satisfied in G). In the general problem of the existence of an inverse element it is necessary to find a single algorithm for all finite-definite semigroups with cancellation. In 1955 S. I. Adyan proved [2] that the general problem of the existence of an inverse element is not solvable, although it is impossible to construct a specific finite-definite semi-lattice with unsolvable particular problem of existence of the inverse element. (Up to then the unsolvability of any general algorithm problem was proved by constructing an example of an unsolvable particular problem. Here this road was closed.) S. I. Adyan gave also an example of a semigroup with an enumerable number of effectively specified defining relations and unsolvable problem of existence of the inverse element.

The methods used to prove the unsolvability of algorithmic problems have permitted S. I. Adyan to clarify the role of the law of cancellation in the specification of semigroups with the aid of defining relations. The

following question was raised: can one specify any finite-definite semigroup with cancellation rule as a semigroup without the cancellation rule (i.e., as an associative system) and furthermore also finite-definite. If one does not require finite-definiteness, then an affirmative answer to this question is obvious. S. I. Adyan [6] constructed a finite-definite semigroup with cancellation rule, which cannot be specified without the rule of cancellation by a finite number of defining relations. It was found that the cancellation rule

$$XA = XB \to A = B$$

cannot be replaced even by a finite number of mixed relations, i.e., relations which can be written only by equality of words, in which together with the formants there are also variables. This result shows that finite-definite subgroups with contraction law cannot be included in the general concepts of finite-definite algebraic systems.

6. The detailed proof published in 1955 by P. S. Novikov of the unsolvability of the problem of identity of the theory of groups leans on the result of Turing on the unsolvability of the problem of identity for semigroups with cancellation.[1] Since the Turing paper with this result was carelessly written and contained incorrect lemmas, the question was raised of a proof of the theorem of P. S. Novikov without using the indicated result by Turing. P. S. Novikov used the result of Turing only to construct a semigroup with one-sided cancellation and an unsolvable identity problem, on the basis he then constructed a group with unsolvable identity problem. In 1957 P. S. Novikov and S. I. Adyan[2] constructed jointly an example of such a semigroup based not on the result of Turing, but on the result of A. A. Markov and of Post on the associative system with unsolvable identity problem.

P. S. Novikov says that the semi-group G is representable by means of group F, if one can separate in F a

1. A. M. Turing, The Word Problem in Semi-Groups with Cancellation. Annals of Mathematics, 52 (1950), 491 -- 505.
2. P. S. Novikov and S. I. Adyan. The Problem of Identity for Semi-Groups with One-Sided Cancellation. Z. F. math. Logik and Grundl. d. Math. (1958), Vol. 4, 1 -- 24.

subset of elements F', which is algorithmically reduced in a mutual-unique correspondence with a set of elements of the semi-group G in such a manner, that equal elements in G correspond to equal elements in F and vice versa. In his paper [30] he proved also the following theorem, which is of independent interest.

For each finite-definite semi-group G there exists a finite-definite group F, with the aid of which it is possible to represent the semi-group G.

7. In spite of the unsolvability of the general problem of identity for semi-groups and for groups, it is of interest to solve the problem of identity for various classes of groups, semi-groups, etc.

As early as in 1947 V. A. Tartakovskiy proved that if the left halves of the defining relations of a finite-definite group "do not superpose one on another strongly" then the problem of identity for this group is affirmatively solved. The degree of superposition of the defining relations is characterized by Tartakovskiy in his paper [26] with the aid of the property of k-cancellability of the basis of the group (and the greater the superposition, the smaller the natural number k), and he solves the problem of identity for groups with a k-cancelled basis for all $k > 6$.

In 1955 A. I. Mal'tsev [42] published an algorithm that solves the problem of identity for nilpotent groups. In 1957 he solved a more general problem for nilpotent groups. He proved that for any nilpotent group it is possible to map homomorphically in a finite group in such a way that two arbitrary prescribed subgroups of the considered nilpotent group, intersecting only in a single element, are mapped into subgroups of a finite group, also intersecting in only a single element.

The problem of identity for Abel groups was solved simply, and the corresponding algorithm was known for a long time. The commutative semi-groups with cancellation are embedded in the commutative groups, from which one obtains an algorithm for them, too. For commutative semi-groups without the law of cancellation, the problem of identity was solved simultaneously in 1956 by G. S. Tseytin and by a student of A. I. Mal'tsev, V. A. Yemelichev.

8. The problem of entry for a given group F is

formulated in the following manner: it is necessary to find such an algorithm, which would permit for each semi-group, generated by a finite number of elements of group F and arbitrary element A of group F, to recognize whether A belongs to this sub-group or not. The problem of entry is formulated also in weak form, when one seeks for each subgroup of group F its own algorithm. From the solvability for any one group of the problem of entry even in the weak form follows the solvability of the problem of identity for this group.

On the basis of the aforementioned paper [42] of A. I. Mal'tsev, it is easy to construct an algorithm that solves the problem of entry for nilpotent groups.

Many problems connected with the problem of entry were raised by P. S. Novikov. His student, K. A. Mikhaylova, is now engaged on these problems. On the basis of the results of P. S. Novikov she has proved that for a direct reproduction of two free groups with two formants the problem of entry is not solvable in weak form. This result was used by A. A. Markov to prove the unsolvability of the problem of representability (see above, Section 2) for matrix of order $n \geqslant 4$. K. A. Mikhaylova proved that the problem of entry for a direct product of an Abel group and a group with solvable problem of entry is solvable. She also established that it is impossible to prove the unsolvability of the weak entry problem for a direct product of a group with a solvable weak entry problem and another group with a solvable entry problem, each subgroup of which can be specified by a finite number of forming elements and defining relations. The foregoing results were reported by K. A. Mikhaylova in 1957 at the MGPI Seminar and at seminars on mathematical logic and on algebra at the Moscow University.

9. Many necessary and sufficient conditions of solvability of the problem of identity for universal algebras was given by A. V. Kuznetsov.

An algebra is called finite-generated if it has a finite number of formants and operations. A. V. Kuznetsov calls a finite-generated algebra finite-definite, if it is specified by a finite number of defining relations of general type (i.e., relations in which together with the formants there participate also variable symbols).

He proved (MGPI Seminar, November 1956) that for solvability of the problem of identity in a finite-generated algebra $\mathfrak{A}$ it is necessary and sufficient that this algebra be covered[1] by a certain simple finite-definite algebra $\mathfrak{B}$. (Under simple algebra is meant here an algebra which has no non-trivial homomorphisms.) The consequence of this theorem is the solvability of the problem of identity for simple finite-definite groups. A. V. Kuznetsov gave for this particular case a very simple algorithm.

The second necessary and sufficient condition of solvability of a problem of identity for finite-generated algebras, found by A. V. Kuznetsov is formulated in terms of the concepts of the general-recursive algebra. A. V. Kuznetsov calls an algebra general-recursive if it is either finite, or is isomorphic to a certain algebra, the elements of which are natural numbers, on the operations of which are general-recursive functions. He proved (MGPI Seminar, March 1955) that for solvability of the problem of identity in a finite-generated algebra it is necessary and sufficient that it be general-recursive. Still another necessary and sufficient condition can be found in the note by A. V. Kuznetsov [3].

With the aid of methods, based on the close relationship between the concepts of finite-definite algebra and general-recursive function, A. V. Kuznetsov proved that it is impossible to construct a "partial algorithm," which would give a complete solution of the problem of identity for any such algebra, for which the problem of identity is solvable (here one does not require at all that the "partial algorithm" solve the problem for a given algebra, with a solvable identity problem or not.)

10. Two invariant group properties α and β, which cannot be satisfied simultaneously in one in the same finite-definite group, are called recursive separable, if there exist such an algorithm, which upon specification of any finite-definite group F yields one of the answers, positive or negative, whereas if in group F the

---

1. One says that an algebra $\mathfrak{B}$ covers an algebra $\mathfrak{A}$ if the sets of their elements coincide and any operation of the algebra $\mathfrak{A}$ is also an operation of algebra $\mathfrak{B}$.

the property α is satisfied, then the answer is affirmative, and if β is satisfied, then the answer is negative. A. V. Kuznetsov, introducing this concept, indicated several simple examples of recursively separable properties. The unitarity and infiniteness are, as he showed, recursively non-separable properties.

Groups F and G are called constructively different if the property "to be a group, isomorphic to group F" and "to be a group, isomorphic to group G" are recursively separable. This concept was introduced by A. A. Markov (Seminar on Mathematical Logic at the Moscow University, February 1957) who advanced the hypothesis that any two groups, coinciding with their commutant, are constructively non-distinguishable. A. V. Kuznetsov proved that a non-Abelian group of eighth order and the results of its Abelization (i.e., addition of the commutability of all pairs of elements to their relations of this group) are constructively distinguishable.

10. Constructive Interpretation of Mathematical Formulations. Constructive Mathematical Analysis.

1. In all-modern mathematical thinking, an important position is occupied by the difference between "constructive" and "non-constructive." Recently among the objects investigated by the mathematics there have been coming to the forefront "constructive-definable objects," examples of which are the natural and rational numbers, words in a certain alphabet, and other objects, which, roughly speaking, can be constructed in a finite number of steps and presented for examination. There exist different points of view with repect to the methods which are admissible in a study of constructive-definable objects. One of these consists of admitting, in the study of constructive objects, any means used in mathematics. Another admits only specific constructive methods. A study of constructive objects by special constructive methods is the subject of the constructive trend in mathematics. In the USSR this trend began to be developed in Leningrad, in the school of A. A. Markov.

The abstractions that serve as the basis for modern constructive trend in mathematics were indicated by A. A. Markov in his article [41]. These include the abstraction of identification, which permits to identify "identical" constructive objects, and the abstraction of potential realizability, which permits imagining constructive objects of as large a "dimension" as convenient. These do not include the abstraction of actual infinity, and therefore within the framework of the constructive direction one delineates a sphere of action of the law of excluded third. Thus, from the point of view of the constructive trend a study of constructive objects requires its "constructive" logic. The principles of such logic were already layed down by Brouwer. In the works of the representatives of the constructive trend, constructive logic receives further development. Thus, A. A. Markov advanced [49, 51], as belonging to the constructive mathematical logic, the following principle, which we shall call the Markov principle: if there is an algorithm that permits ascertaining for any natural whether it has a property C, and if the proposition of non-existence of a natural number with property C has been refuted, then there exists a natural number with the property C. A. A. Markov has shown that this principle is used in proving many mathematical statements, to some of which it is equivalent. In particular, the Markov principle is equivalent to the fact that any formula of the type

$$(\forall x (P \vee \neg P) \supset (\neg\neg\exists x P \supset \exists x P)), \tag{6}$$

is realizable (in the sense of Kleene). Here x is the variable and P is the formula of logic-arithmetic calculus of Kleene. At the same time, as shown by A. A. Markov, the formula

$$(\forall a (A(a) \vee \neg A(a)) \supset (\neg\neg\exists a A(a) \supset \exists a A(a)))$$

is not derivable in the purely intuitionistic calculus of predicates.

The problem of constructive understanding of mathematical opinions concerning constructive objects was raised already by Brouwer, who advanced the principle of

constructive understanding of the opinion on the existence of constructive objects and disjunctions. But no developed sufficiently complete theory of constructive interpretation of mathematical opinions was proposed by him. The first important step in this direction was made by A. N. Kolmogorov [34], who developed the semantics of the theory of problems formulated by means of logical connections of calculus of formulation, and established that the logical formulas, derived in the intuitionistic calculus of formulation, expressed types of these problems, which admit of a constructive solution. The ideas of A. N. Kolmogorov were refined and developed (on the basis of an exact concept of arithmetic algorithm) by Kleene.

Each constant logical-arithmetic formula is considered by Kleene as an incomplete information on the solvability of a certain constructive problem; the interpretation of the formula lies indeed in displaying this problem. N. A. Shanin [15, 17], in criticizing the rules proposed by Kleene for interpretation, advanced new principles of constructive understanding of mathematical opinions. According to N. A. Shanin not all opinions are naturally considered as information concerning the solvability of a constructive problem. Opinions, in which the logical connections $\vee$ and $\exists$ do not participate are not considered in this manner, and understood on the basis of extrapolation of a definite part of classical logic. As concerns those opinions, which contain constructive problems, rules are proposed (different from the Kleene rules) for displaying such a problem.

2. The separation of constructive objects from among the subjects studied in mathematical analysis, and a study of these constructive objects by constructive methods, is the subject of constructive mathematical analysis (see N. A. Shanin [14]. In classical mathematical analysis a real number is defined as a fundamental sequence, considered with accuracy to equivalents of rational numbers. In the constructive mathematical analysis this concept is constructivized. First, only computable sequences of rational numbers are admitted, i.e., sequences for which there exists a computable function, which gives the n-th term from its number n. Secondly,

from among these sequences one chooses only the computably-converging ones, i.e., those having a computable regulator of convergence (the function f is called a regulator of convergence for a sequence $\{a_n\}$, if from the inequality $k \geq f(i), l \geq f(i)$ follows the inequality $|a_k - a_l| < 2^{-i}$); hereone can restrict oneself only to regularly-converging sequences, i.e., those for which from the inequalities $k \leq l$ there follows the inequality $|a_k - a_l| \leq 2^{-k}$. (as is known,[1] not only monotonic and bounded computable sequence of rational numbers is computably-converging.)

Thus, the constructive (computable) real number is defined by A. A. Markov [49] as considered with accuracy to equivalents computable and regularly-converging sequence of rational numbers. I. D. Zaslavskiy [6] proposed two other variants of the concept of constructive real number, starting with the concepts that he introduced for a constructive Dedekind cut and a constructive successively contracting integral; all three variants of the introduction of constructive real numbers were found to be constructively-isomorphic.

For constructive real numbers one defines relations of a quality and order, and also of the action on them. One introduces the concept of computable and fundamental sequences of real numbers, limits, etc. There arises a unique "constructive continuum" which differs in its properties from the ordinary classical continuum (see N. A. Shanin [14]). In particular, as shown by I. D. Zaslavskiy [3, 5] there exists such a constructive sequence of finite sets of segments (with rational ends) Φ_m, that:

1) $\Phi_1 \subset [0, 1]$;
2) $\Phi_1 \supset \Phi_2 \supset \Phi_3 \supset \ldots$;
3) There exists no constructive real number, contained in all Φ_m.

A. A. Markov [49] introduced the concept of a computable or constructive function of real variable. To each computable real number one can assign Goedel numbers that define this number of regularly-convering computable sequences of rational numbers. Any such number we shall

1. E. Specker, Nicht Konstruktivbeweisbare Sätze der Analysis. Journal of Symbolic Logic 14, No. 3 (1949), 154 -- 158.

agree to call as the writing of the considered real number. A partially recursive function of one variable is called single-valued on the given computable real number, if it first is definite for any notation of this number, secondly, it converts each such notation into a notation of a certain constructive real number, and thirdly it converts all such notations into notations of one in the same number.

Any partially-recursive function χ generates, obviously, the function of one computable real variable, defined for those numbers, on which χ is single valued; functions so specified were called by A. A. Markov constructive or computable.

In the same paper [49] A. A. Markov introduced the concept of constructive discontinuity and has shown that the constructive function of real variables does not have a constructive discontinuity at any point. Later on G. S. Tseytin [7] proved that any constructive function defined on all constructive real numbers from an arbitrary interval, is continuous in the constructive sense on this interval.

Already from these results by A. A. Markov and G. S. Tseytin it is seen that the classical and constructive functions of real variable "behave" differently. Many theorems in this direction belong to I. D. Zaslavskiy [3, 5, 6]. Thus, he established the following theorems: 1) There exists a function, continuous but not bounded on [0, 1]; 2) there exists a function, continuous and bounded on [0, 1], but not having an exact (smallest) upper limit on it; 3) there exists a function, continuous unbounded, but not uniformly continuous on [0, 1], whereas for $\varepsilon = 1$ there does not exist a corresponding $\delta > 0$. 4) There exists a function which is uniformly continuous in the interval [0, 1], which assumes on it all values between 0 and 1, but which does not assume values equal to its upper limit 1. It should be noted that the terms encountered in these theorems such as "continuous," "exact upper limit," "interval [0, 1]," are understood in the constructive sense; thus, for example, the interval [0, 1] is considered to consist only of constructive real numbers.

In their paper [4] I. D. Zaslavskiy and G. S.

Tseytin established the existence of a function which is not integralable for any of the generalization integrals that they define. In paper [5] I. D. Zaslavskiy introduced the constructive analogues of the concept of a function of bounded variation and of an absolutely continuous function and investigated their properties.

G. S. Tseytin [1, 5] considered constructive analogues of the following theorems and mathematical analyses: 1) Any sequence of closed intervals included in each other has a common point; 2) the first Cauchy theorem (on the vanishing within an interval of a continuous function, assuming different signs on its ends); 3) the second Cauchy theorem (that a function, continuous in a closed interval, assumes there all the values intermediate between its values on the end); 4) the Rolle theorem, 5) the Lagrange theorem (the finite-increment formula).

The formulations of all these theorems have a certain similar appearance. In fact, it is stated in each of these that for any object of a defined type there exists a real number, which is in a definite relation to this object. If one reinterprets all these theorems constructively, wherein not only the objects themselves (numbers, functions, etc.) are replaced by corresponding constructive objects, but the logical connections are also constructively interpreted, then each theorem becomes a statement of the existence of an algorithm, which gives from the writing of a certain object the writing of a constructive real number. However, as shown by G. S. Tseytin, for all the five statements obtained such algorithms are impossible. With this, for the Cauchy and Rolle theorem of included segments such algorithms will be possible, if one changes the concept of writing of a constructive real number in such a way, that in the writing (in the new sense of the word) is contained information on the computable sequence which defines a real number, but not on its regulator of convergence (although it should exist[1]).

In article [14], N. A. Shanin noted a method of constructing a constructive measure theory. The concept

1. "It cannot not exist," in the terms of constructive logic.

proposed by him of a measurable set is not (as in classical mathematics) a specialization of the concept of the set. Measurable sets according to N. A. Shanin do not have any elements! A measurable set is defined as a converging (in a definite sense), computable sequence of finite collections of segments, with rational ends, and the measure of the measurable set is the limit of the measures (naturally defined) of these finite collections. In article [14] there are introduced concepts of sequences of measurable sets and limits of such a sequence and the question of the similarity and differences between constructive and classical measure theory are analyzed; ways are also indicated of developing the concepts of constructive functional analysis.

3. In order to understand the results of the constructive trend, it is not essential to adhere to the point of view of constructive logic. These results, as were seen above, are converted in classical "language" with the aid of the concept of the algorithm and a computable function. V. A. Uspenskiy in the article [13] undertook to attempt to expound several ideas of the constructive trend (using a specific example of the theorem on uniform continuity) from classical positions. In particular, in this article there are refined, on the basis of the concepts of the theory of algorithms, the concepts "functio discreta," "functio mixta," and "function continua," which were introduced as Weyl as early as in 1921. In the article is found a constructive analogue of the concept of continuous function from the point of Baire space J. Each continuous function F can be specified on Baire space by means of a certain function φ, which converts the corteges of natural numbers into corteges, so that if

$$m^{(1)}, m^{(2)}, \ldots, m^{(k)} \ldots$$

is an increasing sequence[1] of segments of the point

$$a \in J, \text{ то } \varphi(m^{(1)}), \varphi(m^{(2)}), \ldots \varphi(m^{(k)}), \ldots$$

1. That is, each successive cortege is a continuation of the preceding one.

is an increasing sequence of segments of the points

$$\beta = F(\alpha).$$

If φ is a computable function, then the function F it generates is called constructively continuous (a constructively continuous function, considered on computable points, is taken to be as the constructive analogue of a continuous function). In a similar manner the concept of constructive-continuous function is also introduced for the generalized Baire space (see Section 6 of the present survey). The points of the generalized Baire space can be visualized as functions that are defined on subsets of a natural series, and consequently, a function on a generalized Baire space can be visualized as an operator on a system of such functions. From the earlier results of V. A. Uspenskiy [6] there follows the theorem, formulated by him in [13], that a function on a generalized Baire space (with values from the same space) is a computable operator when and only when it is constructively continuous. This circumstance makes it possible to obtain the Kuznetsov-Trakhtenbrot theorem on the region of general-definiteness of a computable operator. In turn, the constructive analogues of principal topological concepts introduced by A. V. Kuznetsov [2] and B. A. Trakhtenbrot [5], i.e., concepts of the effectively open, effectively closed, etc., sets, is widely used in article [13]; a simpler example, than that given by Kleene, is constructed for a function which is constructively continuous on a constructive compact, but is not uniformly continuous on it.

Chapter IV

LOGICAL AND LOGICAL-MATHEMATIC CALCULI

11. Constructive Calculi from the Classical and Constructive Points of View.

1. The constructive-logical and logical-arithmetic calculi have attracted the attention of Soviet mathematicians as early as in the Twenties of the current century. The well-known works of A. N. Kolmogorov [8, 34] and V. I. Glivenko [4, 5], devoted to the clarification and the relations of classic and constructive logical and logic-arithmetic calculi. (Under constructive we understand here those formalizations proposed by A. N. Kolmogorov, V. I. Glivenko and A. Heyting of "intuitionistic" arithmetic and logic,[1] which makes no use of the law of excluded third as applied to finite sets of objects, i.e., which does not admit of abstraction of actual infinity.) In reference [8] A. N. Kolmogorov first proposed such an interpretation of the derivable formulas of classical arithmetic, under which they are transformed into derivable formulas of constructive arithmetic, i.e., into formulas, in the derivation of which one does not admit the application of excluded third. In [34] A. N. Kolmogorov gave, to the contrary, an interpretation of the Heyting logical calculus as a calculus of problems (and not propositions) of ordinary ("classical") mathematics.

---

1. As was already noted (see introduction), Soviet mathematicians and logicians consider the use of the terms "intuitionistic logic" or "intuitionistic arithmetic" by persons who are far from philosophy of intuitionism as introducing confusion, and incorrect. The calculations of A. N. Kolmogorov and V. I. Glivenko were proposed indeed in order to separate the specific results, obtained in the school of "intuitionists" founded by Brouwer, from his intuitionistic philosophy. Things are different, naturally, for the representatives of the Heyting intuitionism.

At the end of the Thirties P. S. Novikov ([17], published in 1942) proposed a proof of non-contradiction of classical (i.e., making free use of the laws of excluded third) arithmetic, based on a certain extention of constructive principles of "intuitionistic" mathematics to logical sums (products of a denumerable number of components (factors). In all these directions, work was continued also in the period of interest to us now. In the school of A. A. Markov particular attention was paid to the development of constructive logical and logical-mathematical calculi.

2. In interpreting the Brouwer logic as calculus of problems, A. N. Kolmogorov [34] did not dwell on the questions of what is a "problem," what does it mean "to solve an elementary problem," what does it mean "to reduce a solution of problem A to a solution of problem b," in what meaning is the "reduction of problem A to problem B" can be considered in turn as a "problem"? Some of these questions concerning the relations between the provability in the Heyting calculus (which was directly interpreted by A. N. Kolmogorov) and solvability as applied to problems it was difficult to answer. In the development of the ideas of A. N. Kolmogorov, S. C. Kleene[1] proposed a special method of "realization" of logical-arithmetic formulas, which has, as shown by D. Nelson[2] that property that any formula, provable in constructive arithmetic, is "realizable" in the sense of Kleene. (The inverse is found to be untrue even for the calculus of projections[3].) A student of A. N. Kolmogorov, Yu. T. Medvedev, proposed in

1. S. C. Kleene. On the interpretation of Intuitionistic Number Theory. Journal of Symbolic Logic, 10, No. 4 (1945). 109 -- 123, see also S. C. Kleene, Introduction to Meta-Mathematics, Moscow, Foreign Literature Press, 1957, Section 82.
2. D. Nelson, Recursive Functions and Intuitionistic Number Theory. Transactions, American Mathematical Society, 61 91947) 307 -- 368.
3. Concerning this result by Gene Rose, see at the end of item 4 and in item 9.

his dissertation[1] (see also remarks [5, 6] a second refinement of the above-mentioned concepts and expressions as applied especially to "mass problems" (in the sense of Medvedev) and their "algorithmic solution."

Inasmuch as the concepts of the mass problem in the sense of Medvedev we have already dwelled on in Section 6, it is enough to mention here only that since a partially ordered set Ω of degrees of difficulty is, as shown by Yu. T. Medvedev, an implicative structure,[2] any class Ω^e of mass problems $a \leqslant e$, where e is a certain fixed degree of difficulty, is found to be an exact interpretation of constructive logic (Dissertation, theorem 4). With the aid of this interpretation it is possible to compare with each logical-arithmetic formula a certain mass problem the degree of difficulty of which characterizes the "degree of non-constructiveness" of the prediction stated by this formula. If we give the name of "effectively true" to those logical-arithmetic propositions, to which correspond the solvable mass problems, we obtain a new definition of constructive truth in arithmetic as proposed by Yu. T. Medvedev in his dissertation.

3. A student of A. A. Yanovskaya, B. Yu. Pil'chak, engaged in constructive calculi equivalent to the calculus of V. I. Glivenko. In her dissertation[3] and in [3], B. Yu. Pil'chak gave first of all a general characteristic of this class of calculi, as having seven properties which she listed, of which we shall note the following: 3) -- 4) the connection between the implication and the derivability (theorem on the deduction and modus ponens); 6) the provability of the disjunction $A \vee B$ when and only when at least one of the formulas A or B is provable; 7) provability of negation $A(\neg A)$ when and only when the premise of the provability of formula A leads to a contradiction;

---

1. Yu. T. Medvedev, Degrees of Difficulty of Mass Problems, Dissertation, Moscow State University, 1955.
2. In other words, structure, dual to a structure with relative pseudo-complements (See Birkhoff, Lattice Theory, Moscow, Foreign Literature Press, 1952, p. 273).
3. B. Yu. Pil'chak, On the Calculus of the Problems of A. N. Kolmogorov, Dissertation, Moscow, 1950.

2) provability of any formula C under the condition that any false formula is provable. The description given by B. Yu. Pil'chak is complete, inasmuch as[1] any calculus that has the property 1) – 7) is equivalent to the calculus of Glivenko.

As is well known (this is proved by Goedel as early as in 1930) there exists no finite matrix (set of values of truth values on which truth functions corresponding to logical couplings are defined) in which the "identically true" (assuming only separated values for any distribution of truth values with respect to the variables that enter in them) formulas were not only those which are proved by the Heyting calculus of predictions. In 1936 the Polish mathematician Jaskowski constructed a sequence of finite matrices $\{I_\chi\}$ such that the formula A is proved in the Heyting calculus if and only if it is "identically true" in all I_n. The proofs of his theorem, just like the proofs of the theorem on the canonical ("regular") representation of formulas proposed by him (with accuracy to deductive equality in the Heyting calculus), was not published by Jaskowski. In her dissertation (May 1950) B. Yu. Pil'chak not only proved Jaskowski's propositions, but also found such a proof (published in [3]) which permitted her to construct a much simpler algorithm than that proposed earlier (by Gentzen and Wajsberg), one which solves the problem of solvability for the Heyting calculus.

4. In 1954/55 Academic Year, P. S. Novikov delivered at the Moscow University a course of lectures on constructive logic, which is presently being readied for print. The basis of the course was the classic calculus of predictions (and later also predicates), broadened by adding to it the provability operator, defined by certain axioms and a rule of deduction. The calculus thus obtained was called by P. S. Novikov B-calculus. Already in 1931 Goedel stated a guess, which was later proved by many authors, that a calculus of this kind can serve as an exact classic interpretation of intuitionistic logic (for example, the Heyting calculus). ("Exact" in that sense, that any formula A is provable "intuitionistically"

1. B. Yu. Pil'chak, On the Calculus of the problems of A. N. Kolmogorov, Dissertation, Moscow, 1950.

when and only when its "translation" A' into the language of the D calculus is provable in the latter.) In his course, P. S. Novikov gave a new simple proof of an analogous statement and constructed an arithmetic model of D calculus, corresponding essentially to the situation which arises during the measurements of quantities, which are never realizable in practice with absolute (ideal) accuracy. (The elementary predicates in this model have the form «$f(x_1, x_2, \ldots, x_n) < g(x_1, x_2, \ldots x_n)$» where f and g are linear arithmetic functions; $D\mathfrak{A}(x_1, x_2, \ldots, x_n)$ is true at the point $(x_1^0, \ldots, x_n^0)$, if there exists a vicinity of this point, in which $\mathfrak{A}(x_1, x_2, \ldots x_n)$ is true.)

Great attention was paid in the course to topological models of D calculus and to the Heyting calculus. To prove the topological completeness of the Heyting calculus one constructs a space of deductive chains, in which the topology is introduced by specifying elementary open sets $O\mathfrak{A}$, which are treated as an aggregate of deductive chains, containing the formula $\mathfrak{A}$. It is proved that if $\mathfrak{A}$ enters in all the deductive chains, then it is provable in the Heyting calculus. Later on the space of all the deductive chains is replaced by a similar space, homeomorphic to Baire space. On the basis of D calculus of predicates, there was constructed in the course an arithmetic (recursive functions were introduced together with the concept of the Goedel numbering, and the incompleteness theorem was proved). The course was completed with an examination of problems connected with the realization in the sense of Kleene. In particicualr, a constructive proof was given for the theorem of Gene Rose[1] on the constructive incompleteness of the Heyting calculus of prediction (i.e., that not every "realizable" formula (in the sense of Kleene) in the calculus of prediction is provable in the calculus of Heyting), and the existence was also proved of an arithmetic formula which expresses realizability.

1. G. F. Rose. Propositional Calculus and Realizability. Transactions, American Mathematical Society, 75 (1953), 1 -- 19. See also S. C. Kleene, Introduction to Meta-Mathematics, Moscow, Foreign Literature Press, 1957, 454, Translator's comment.

5. While in the school of P. S. Novikov the constructive logic was studied from a broad (not refuting even classical considerations) point of view, in the school of A. A. Markov the abstraction of actual infinity and the corresponding application of the law of excluded third were not admitted even in the investigation of such constructive objects as the formulas or proofs for logical and logical-arithmetic calculi. For illustration of this point of view we give an example pertaining to the constructive interpretation, developed by Kleene and Nelson, of logical-arithmetic formulas with the aid of the so-called "realization" or, in the terminology of A. A. Markov, "filling."[1] The definition of fillability is constructed in such a way that if formula P is fillable, then the formula $\neg P$ (negation of formula P) is not fillable; to the contrary, if formula P is not fillable, then the formula $\neg P$ is fillable. The disjunctions of formulas A and B (i.e., the formula $(A \vee B)$) is fillable if at least one of these formulas is fillable it follows classically from this immediately that the formula $(P \vee \neg P)$ (which expresses the law of excluded third) is fillable for any P, which does not contain three variables. From the constructive point of view, this, however, is not so, for speaking in this manner of any formula P means a consideration of all the infinite sets of formulas P as existing simultaneously, i.e., from the point of view of the abstraction of actual infinity, excluded from the constructive mathematics. To state the existence of fillability for each formula $(P \vee \neg P)$ one can only if a constructive method (algorithm) is possible, comparing with each formula R of the type $(P \vee \neg P)$ a certain filling of the formula R (see N. A. Shanin [13], p. 46); since an algorithm of this kind is impossible, then within the framework of the constructive trend in mathematics it is impossible to advance as a logical principle the thesis that each constant formula of the type $(P \vee \neg P)$ is fillable (N. A. Shanin [13], p. 47).

1. This terminology is apparently connected with the fact that the classical propositions are considered as incomplete communications, which require filling with additional information, which imparts to them an effective character.

6. The principal objects of investigation in the school of A. A. Markov during the period of interest to us were not the logical, but logical-mathematical calculi, the theory of which according to N. A. Shanin (/ 13 /, p. 9) is an earlier chapter in modern mathematical logic, in the theory of logical calculi, since it is closer to mathematics, i.e., it bears a less abstract character. In the present section we shall engage in special constructive logical-arithmetic calculus Σ, obtained by adding to the system of axioms of Peano recursive definitions of addition and multiplication and logical axioms and rule of deduction, equivalent (in their aggregate) to the narrow calculus of predicates of Heyting (the law of excluded third does not figure in this calculus). The addition to the calculus Σ of all possible logical arithmetical formulas of the type of the law of excluded third yields a calculus which is called by N. A. Shanin (/ 13 /, p. 10) the basic classic logical-arithmetic calculus and denoted Σ^{*}. In papers / 11, 12, 13 / N. A. Shanin engaged in the development of methods, proposed by A. N. Kolmogorov / 8 / and by K. Goedel (in 1931) of "submerging" the classical arithmetic (Σ^{*}) in the constructive one (__), i.e., a certain special (indirect) measure of constructive interpretation of the premises of classical arithmetic. We shall dwell on these investigations in greater detail in the next item. Here we shall note that inasmuch as various methods are possible of constructive interpretation of truth (or respectively falsity) of judgement, one can speak of different operations, which juxtapose to the truth (respectively false) judgements constructively, (i.e., in a stronger sense) true (respectively false) ones. One of such operations of constructive falsity was considered in 1949 by D. Nelson.[1]

Ordinary negation of prediction of generality $(\neg(x)A(x))$ in a calculus of type Σ is true, if the proposition that $(x)A(x)$ is true leads to a contradiction. The "constructive" negation derived by Nelson (A. A. Markov / 40 / calls it "strong" negation and denotes it $\sim$) differs in that the prediction $\sim(x)A(x)$ is true if a

1. D. Nelson, Constructable Falsity, Journal of Symbolic Logic, 14, No. 1 (1949), 16 -- 26.

contradicting example is given for the prediction $(x)A(x)$.

In two papers at the Seminar of the Leningrad Division of the Mathematical Institute (LOMI) on 6 and 8 October 1949, A. A. Markov [40] expounded on a logical arithmetic calculus, constructed by him on the basis of the concept of normal algorithm, and including along with ordinary logical operations also the operation of strong negation, and proved various results obtained by him for this operation. In particular, it was found that the strong law of excluded third, i.e., the formula $A \vee \sim A$, can be refuted with an example (it is possible to construct such a formula A, for which $\neg(A \vee \sim A)$ takes place). It was also found that for strong negation in the general case the laws of contraposition are no longer true, and therefore the "principle of spatiality" stops being true (the rule of replacement by an equivalent). Since, however, this rule retains its force for the replacement of P by Q (or, conversely, of Q by P) in those cases when along with $P \equiv Q$ there takes place also $\sim P \equiv \sim Q$, A. A. Markov introduces for such P and Q the concept of complete equivalence: $P \ddot{\equiv} Q$. It is found [40], that a complete equivalent takes place: $\neg P \equiv (P \supset \sim P)$, expressing the usual negation in terms of a strong negation and implication.

7. The logical calculus of predictions Π^+, which is obtained from the Heyting calculus by adding to it the operation of strong negation $(\sim)$, defined by the axioms: $\sim p \supset (p \supset q)$, $\sim(p \supset q) \equiv (p \& \sim q)$, $\sim(p \& q) \equiv (\sim p \vee \sim q)$, $\sim(p \vee q) \equiv (\sim p \& \sim q)$, $p \equiv \sim\sim p$, $p \equiv \sim\neg p$, was considered by a student of A. A. Markov, N. N. Vorob'yev [4]. (When the propositional variables in the proved formulas of the calculus are replaced by arithmetic formulas, one obtains provable formulas of logical-arithmetic calculus of A. A. Markov with strong negation). N. N. Vorob'yev has shown that any formula of the Π^+ calculus can be reduced to such a formula, at which only elementary predictions (letters) remain under the sings of strong negation (if such are generally not excluded. Furthermore, the following theorem concerning normal form holds [4]:

Each formula A of the Π^+ calculus can be effectively represented in the form of a conjunction of formulas $B_1, \ldots, B_k$, such that in not a single B_i contains the

signs $\&$ or $\neg$, only a letter (propositional variable) can be found under the sign of strong negation, not a single B_i has a part of the form $(p \vee q) \supset r$, while the formula $A \equiv B_1 \& \ldots \& B_k$ is itself provable in Π^+ .

In reference [5], Vorob'yev constructs an algorithm, which solves for the Π^+ calculus the problem of solvability, i.e., which permits for each formula A, written in the language of this calculus, to solve effectively the question whether A is provable in Π^+ or not. With the aid of this algorithm it becomes clear, in particular, that the formula $\neg p \supset \sim p$ is not provable in Π^+ . Since the formula $\sim p \supset \neg p$ is provable in Π^+ , then the name of "strong negation" for the operation $\sim$ is found to be fully justified.

Strong negation belongs thus to the number of such operations η , for which the implication $\eta R \supset \neg R$ is constructively justified for any R. It defines, consequently, in the terminology of N. A. Shanin ([13], pp. 80 -- 81) a certain "particular type of concept of constructive falsity" (analogous, constructive justifiability, for example in the sense of fillability, of any formula $\xi R \supset R$ denotes that the operation ξ "defines a particular type of concept of constructive truth."). Furthermore, inasmuch as for elementary formulas (i.e., the equalities T = S, where T and S are terms in the Σ calculus) the operation of strong negation can be eliminated (it is equivalent to simple negation), then it follows from the results of N. N. Vorob'yev that a strong negation actually belongs among those operations η , with which it is possible to compare effectively to each formula R of arithmetic an arithmetic (i.e., not containing the sign of η) formula R', the provability of which in the Σ calculus can be considered as establishment of the constructive falsity of formula R. In papers [11, 12, 13], N. A. Shanin considers (in connection with the problem of submersion of the classical calculus Σ in the constructive calculus Σ^+) several analogous operations, defining particular type of the concept of constructive truth or constructive falsity.

8. The first submersion operation, taken to be meant as an algorithm, applicable to any arithmetic

formula R and convertint it into another arithmetic formula R', and furthermore, in such a way that if R is provable in Σ^*, then R' is provable in Σ (and vice versa), was already considered in 1925 by A. N. Kolmogorov. (Another operation of the same kind was published in 1931 by K. Goedel.) The operation of Kolmogorov consisted of "hanging" two signs of negation on each entry into formula R of its sub-formula (including R itself). This and several other submersion operations, which represent a modification[1] of Kolmogorov's operation, but much more simply realizable (requiring no "hanging" of double negation on any sub-formula of formula R), was considered in note [12], and also in Chapter II of [13] by N. A. Shanin. But the submersion operations θ of this type have, from the constructive point of view, that effect, that the very transition from formula R to formula θR is in the general case not found constructively: there exist such formulas R, for which the implication $R \supset \theta R$ is unfillable. (The existence of such formulas is due to the fact that, for example, the formula $\forall x \neg\neg(\Gamma \vee \neg\Gamma)$ is derivable in the Σ calculus (and, consequently, is fillable) for any formula Γ with one free variable x (therefore negation of such a formula is unfillable); yet there exist such "unsolvable" formulas Γ_0 (of the same type), for which the formula $\neg\forall x(\Gamma_0 \vee \neg\Gamma_0)$ is fillable. If one takes R_0 to be the last formula, then for any submersion operation θ, which converts R_0 into a formula of the type $\neg\forall x \neg\neg(\Gamma \vee \neg\Gamma)$, the premise will be fillable in the implication $R_0 \supset \theta R_0$, but the conclusion will not be fillable. This application, consequently, will not be fillable).

Introducing into consideration certain new operations, which define particular type of concepts of constructive truth and constructive falsity, N. A. Shanin [12, 13] constructed several submersion operations, called by him "regular submersion operations," which do not have this defect.

9. P. S. Novikov in his paper [17] proved that

1. The operation ; according to N. A. Shanin ([13], p. 59), is a modification of the operation , if no matter what the formula R, the formula is derivable in the calculus.

if in classical arithmetic one can prove a formula of the type $\exists x B(x)$, where $B(x)$ expresses a solvable (recursive) predicate, then it is possible to indicate effectively a number n, for which $B(n)$ takes place (so that the formula $\exists x B(x)$ is provable and constructive). (This result permits in many cases to extract constructive, and furthermore elementary proofs from classical proofs). Here use is made of a certain type of induction, which consists essentially of a transition from all the elements of recursive sequence to the sequence itself. A. S. Yesenin-Vol'pin [4] reinforced this result somewhat, obtaining it by weaker means, namely by means of induction to the first ε-number ε_0 . This result also pertains to the result of V. S. Novikov, as a proof of the non-contradictability of the arithmetic belong to Hentzen to the proof of P. S. Novikov (leaving aside the fact that the method of P. S. Novikov, being stronger, makes it possible to justify the methods of transfinite induction, used by Hentzen (1936, 1938) and Schutte (1951) in proofs of non-contradictability). This result can be applied to a constructive proof of the theorem of Gene Rose on the constructive incompleteness of the Heyting calculus of prediction, which was done in 1954 simultaneously and independently of each other by P. S. Novikov and A. S. Yesenin-Vol'pin.[1]

12. Logical Calculi and Their Models. Problems of Solvability, Completeness and Non-Contractability.

1. The subject of the theory of logical and logical mathematic calculi (which we shall call in this section for the sake of brevity "logical" or "deductive" calculi) are very closely related with the theory of algorithms, both ordinary and conditional (algorithms of reducibility or the computable operators corresponding to them). Also, the rules of formation of imaginable ("correct") formulas

1. S. C. Kleene, Introduction to Meta-Mathematics, Moscow, Foreign Literature Press, 1957 (p. 454, remark of translator).

of these calculi, and the rules of deduction (conversions of certain formulas into others) usually have also an algorithmic character. Problems of solvability of such calculi are directly algorithmic problems. Problems on the other hand of completeness are also in a definite relation (in which we shall dwell in detail below) with certain mass problems (problems of separability), i.e., also problems involving the search for an algorithm. Therefore the theory of algorithms finds a direct application in the theory of logical and logical-mathematical calculi. However, conversely, logical calculi are used in algorithm theory. Thus, we already mentioned that a computable function can be defined in terms of a formal derivability of certain formulas of the narrow calculus of predicates. Another refinement of the concept of the algorithm (as an effective computing process) by means of the narrow calculus of predicates was proposed in 1949 by B. A. Trakhtenbrot [1, 2][1]. This definition is based not on the derivability of certain formulas in the calculus of predicates, but a dual one (in a definite sense) of derivability -- interpretability (the existence of a model for performability) of the formula of this calculus. It is known that along with processes of formal derivation, one can indicate also another effective process, applicable to formula of calculus of predicates. Namely, for each such formula Y_A and for each finite set q it is possible to verify whether can be interpreted on the set q or not, i.e., whether Y_A has a model on the set q (B. A. Trakhtenbrot [6], p. 63). In the papers of B. A. Trakhtenbrot [6][1] it is established that the process of simulation on finite classes (sets) and the process of formal derivation are equivalent in that respect, so that they can serve to an equal degree as descriptions of the effective computing processes. Specifically, a definition is given for a function, simulated on finite classes, and it is proved that this concept is equivalent to the concept of

1. See also B. A. Trakhtenbrot, Problem of Solvability on Finite Classes and of the Definition of a Finite Set. Abstract of dissertation, Kiev, 1950.

general-recursive function (B. A. Trakhtenbrot [6]).[1]
The paper [6] contains furthermore complete proof (without leaning on the Robinson theorem used in the article by B. A. Trakhtenbrot [1]).

2. The definition of an algorithm in terms of simulation of functions on finite classes was useful to B. A. Trakhtenbrot in connection with his solution of the principal problem to which his dissertation was devoted: the problem of solvability on finite classes. The problem of solvability for a narrow calculus of predicates (the unsolvability of which was proved in 1936 by Church) can be formulated as a problem of finding an algorithm, which recognizes from the form of the formula $\mathfrak{A}$ whether it is identically true in any object region (finite or infinite) or not. From the non-existence of an algorithm for such a problem there still does not follow the non-existence of such an algorithm for the case when the sought algorithm should recognize only whether the formula $\mathfrak{A}$ is identically true in any finite region. To the contrary, it may even be found that in such a statement of the problem (we shall call it the problem of solvability on finite classes) the question perhaps is answered in the affirmative.

For formulas of a certain particular type, Ackermann solved the problem of solvability on finite classes. I. I. Zhegalkin [8] simplified the Ackermann method and strengthened its result, by proposing an algorithm

1. In explaining the abstracts given here we note that one says of a function f (for convenience of exposition, this function is assumed to be single-placed one says that it ([6], p. 66) is simulated by a formula $\mathfrak{A}(M, N, \ldots)$ (where M and N are single-placed predicates variables) if: a) in each finite model of the formula $\mathfrak{A}$, in which the cardinality of the volume of the predicate M_+ (placed in lieu of M) is equal to m, and the cardinality of the volume of the predicate N_+ (placed in lieu of N) is f(m); b) for each natural number m there exists such a finite model of the formula $\mathfrak{A}$, in which the cardinality of the volume M_+ is equal to m. Thus, the function f(m) = m + 1 is simulated by the formula

$$\exists x (\overline{M(x)} \& N(x) \& \forall y (y \neq x \to (N(y) \equiv M(y))))$$

which solves the problem of solvability on finite classes for formulas of the type

$$\exists x \forall y F(x, y) \vee \exists z_1 \exists z_2 \ldots \exists z_n \mathfrak{A}(F, \Phi_1, \Phi_2, \ldots, \Phi_p; t_1, t_2, \ldots, t_n), \qquad (7)$$

where $F, \Phi_1, \Phi_2, \ldots, \Phi_p$ are two-place predicate variables. Since Ackerman earlier reduced (even!) the general problem of solvability of the narrow calculus of predicates to a similar one for formulas of the type

$$\exists x \forall y F(x, y) \vee \forall z \exists z_1 \exists z_2 \ldots \exists z_n \mathfrak{A}(F, \Phi_1, \Phi_2, \ldots, \Phi_p; z, t_1, t_2, \ldots, t_n) \qquad (8)$$

($F, \Phi_1, \Phi_2, \ldots, \Phi_p$ are the same here as in (7)), then to solve the problem of solvability on finite classes it remained, so to speak, to make one more step, generalizing the result of I. I. Zhegalkin. However, it was impossible to realize this step. As noted in dissertation of A. A Zykov,[1.] the proposition of unsolvability of the problem of solvability on finite classes was advanced in 1949 by P. S. Novikov, who remarked that were the sought algorithm available, it would permit also to solve problems analogous to the Fermat problem (for example the following: construct an algorithm which recognizes for any natural number n, whether there exists (a non-vanishing)natural number x, y, z, such that $x^n + y^n = z^n$). A student of P. S. Novikov, B. A. Trakhtenbrot [1][2.] proved at the end of the same year, 1949, the correctness of this hypothesis of P. S. Novikov,[3.] i.e., the non-

1. A. A. Zykov. On the Reduction of the Problem of Solvability in Logical Calculi. Dissertation, Moscow State University, 1950.
2. See also B. A. Trakhtenbrot, Problem of Solvability on Finite Classes in the Definition of a Finite Set. Abstract of dissertation, Kiev, 1950.
3. In his proof, B. A. Trakhtenbrot [1] used the possibility of effectively constructing for each general-recursive formula f (of one argument) a simulating formula $\mathfrak{A}(M, F, \ldots)$. The point is that the equation f(m) = 0 has an integer non-negative root if and only if the formula $\mathfrak{A}(M, F, \ldots) \& \forall x \overline{F}(x)$ has a finite model, i.e., when the negation of this formula is not identically

existence of an algorithm for the problem of solvability on finite classes ([1], theorem 2).

From this result it was easy to obtain furthermore that the class K_ω of formulas, which are identically true in any finite region, does not have the property proved by W. Wajsberg[1] for classes K_n of formulas, which are identically true in the region containing n elements. Namely, while both the joining to the axioms of the narrow calculus of predicates of any formula from K_n does make any formula of class K_n derivable, there exists for any formula $\mathfrak{A}$ from K_ω such a $\mathfrak{B}$ in K_ω that $\mathfrak{B}$ is not derivable even after the joining of formula $\mathfrak{A}$. We already discussed the expansion of this and other results to a broad class of axiomatic theories of sets, and also to the related results of the question of the equivalence of two definitions of a finite set, in Section 1.[2] We shall have occasion to stop later on on certain aspects pertaining to the proof obtained in this manner by B. A. Trakhtenbrot [2, 11] of the deductive incompleteness of formalized systems of the theory of sets.

Footnote (3) (cont.) from pg. 135. ...true in any finite region. By constructing an algorithm for the problem of solvability on finite classes, we would thus obtain the possibility of extracting from it an algorithm, which recognizes whether an equation of the form f(m) = 0 (where f is arbitrary (single-place) general-recursive formula) has an integer non-negative root or not. But the impossibility of such an algorithm was proved by Church as early as in 1936.

1. W. Wajsberg, Math. Ann. 109 (133).

2. We note here still another proof, obtained by B. A. Trakhtenbrot (see Dissertation) in passing (independently of other authors), for the proposition made by Hilbert and Bernays (D. Hilbert and P. Bernays, Grundlagen der Mathematik, Vol. 1, p. 124) concerning the deductive independence in calculus of predicates of the following two formulas, which are identically true in any finite region:

$$(\mathfrak{F}) \quad (x)(Ey)R(x,y) \,\&\, (x)\overline{R(x,x)} \rightarrow (Ex)(Ey)(Ez)(R(x,y) \,\&\, R(y,z) \,\&\, \overline{R(x,z)}),$$

$$(\mathfrak{G}) \quad (x)(Ey)F(x,y) \,\&\, (Ey)(x)\overline{F(x,y)} \rightarrow$$
$$\rightarrow (Ex)(Ey)(Eu)(Ev)(F(x,u) \,\&\, F(y,u) \,\&\, F(v,x) \,\&\, F(v,y))$$

-1-

3. The problem of solvability on finite classes was engaged also by another student of P. S. Novikov, A. A. Zykov.

In connection with the fact that the algorithm for the problem of solvability of the narrow calculus of predicates is impossible, particular significance attaches to the effective reduction of the problem of fulfillability of the formula A of the general type to the problem of fulfillability of another formula B of a special type. Leaning on the results of Goedel, Loewenheim, and Ackermann, the Hungarian mathematician Kalmar and his student Suranyi gave methods for reducing (in this sense) the formulas of the narrow calculus to any of the following forms:

$$(x_1)(x_2)(x_3)Ey_1 \ldots Ey_n \mathfrak{A}(F;\, x_1, x_2, x_3, y_1, \ldots, y_n), \tag{9}$$

$$(x_1)(x_2)(x_3)Ey(x_4) \ldots (x_n)\mathfrak{A}(F;\, x_1, \ldots, x_n, y), \tag{10}$$

$$(x_1) \ldots (x_n)Ey\mathfrak{A}(F;\, x_1, \ldots, x_n, y), \tag{11}$$

where F is a two-space predicate variable.

By virtue of the result of B. A. Trakhtenbrot, there exists no algorithm also for the problem of solvability on finite classes, and therefore the problems of reduction of the formula in the same sense as for the fulfillability in a finite region, has the same significance, as in the case of the fulfillability in general. As shown by A. A. Zykov, the reduction to the form (9), (10), and (11) is possible in this case, too. Suitable changes in the arguments of Goedel, Loewenheim, Ackermann, Kalmar and Suranyi allow us to obtain theorems for the reduction in the following general formulation: for any formula A of the narrow calculus (with the identity predicate) it is possible to construct effectively a formula B of any of the forms (9), (10), or (11) such that both formulas simultaneously are either not fulfillable, or else are fulfillable only in an infinite region, or are fulfillable in finite regions, with the cardinality of the smallest of the fulfillable regions for formula P being expressed by a primitive-recursive function of analogous cardinality and constructive parameters of the formula A.

By virtue of the Loewenheim theorem, any function

of the narrow calculus of predicates (with identity) is either performable in any infinite region, or else is not fulfilled in any infinite region. Therefore it is possible without risk of falling into set-theoretical antinomies, to define a spectrum[1] of such a formula as an aggregate of all cardinalities $\leq \aleph_0$ of these regions, on which the formula is fulfilled. In the broadened calculus of predicates (of second degree) there is no analogue of the Loewenheim theory, and a naive definition of the spectrum of the formula cannot be acceptable. At the same time for a concrete quantitative (i.e., one not containing free variables) formula and a concrete cardinal number τ it is meaningful to question whether a given formula is fulfilled in a region of cardinality , meaning, the question of a correct definition of a spectrum of a formula of broadened calculus is not removed. Not having as yet such a definition, one can nevertheless carry out transformations of formulas which do not change their "spectra" or which change them in an observable manner, A. A. Zykov [2][2] proved that for any quantitative formula A it is impossible to construct effectively such a formula B, that if A is true in the region of cardinality τ, then B is true in the region of cardinality $\tau + \tau^{l} + 2^{\tau^{l}}$ (l is the number of places of the most multiple-spaced predicate in A), and vice versa, if B is true in the region of cardinality σ, then σ has the form $\tau + \tau^{l} + 2^{\tau^{l}}$, where τ is another cardinal number, less than σ, and A is true in the region of cardinality τ; with this, B has the form

---

1. Earlier, in note [1], B. A. Trakhtenbrot defined the related concept of a spectrum (single-place) of a predicate M, entering in formula $\mathfrak{A}(M, N, \ldots)$ as a subset C of a natural series such that $m \in C$ when and only when there exists a finite model of the formula $\mathfrak{A}$, in which the cardinality of the volume of the predicate M_+ (see footnote at the end of item 1) is equal to m.
2. See Also A. A. Zykov, On the Reduction of the Problem of Solvability in Logical Calculi, Dissertation, Moscow State University, 1950.

$$E\,\mathfrak{W}(Q)\langle\{y\}\rangle\mathfrak{B}(\mathfrak{W}, Q; \{y\}). \qquad (12)$$

where $\mathfrak{W}$ is a two-place and Q a one-place predicate, while $\langle\{y\}\rangle$ is a sequence of quantors in the object variables of the aggregate $\{y\}$. It follows from this that when searching for a sensible definition of the spectrum it is enough to consider only formulas of type (12). From the same theorem one obtains a new proof of the result of Ackermann concerning the unsolvability of the problem of exclusion, namely: if one takes for A to be any formula of the form

$$E P_1 \dots E P_k \langle\{x\}\rangle \mathfrak{A}(P_1, \dots, P_k; \{x\})$$

(i.e., in a certain sense a formula of narrow calculus, fulfillable only in an infinite region, and one constructs from it a formula B of the form (12), then it is impossible to exclude Q from the latter (or in the opposite we would have obtained the formula of narrow calculus, which is fulfillable only in a non-denumerable region, which is impossible).

To reduce the formulas in the sense of equivalence (which also does not change the "spectrum" in the case when the formula is quantitative) A. A. Zykov obtained the following results: a) any formula can be reduced to a beforehand specified form in such a way that the quantors in the predicates proceed the quantors in the objects: b) the quantor prefix of the form

$$\dots (P_1)(P_2) \dots (P_k) EQ_1 EQ_2 \dots EQ_l (R_1)(R_2)(R_3) \dots (R_m) \dots \langle\{x\}\rangle$$

can be reduced to the form

$$\dots (P) EQ (R) \dots \langle\{y\}\rangle$$

with the same number of successive variables of the type as in the quantors in the predicates.

4. In treating the work of A. A. Muchnik [2] in Section 8, item 2, we already spoke of his results, which establish the connection between the problems of non-solvability and non-separability. Certain problems of the same kind were considered earlier (1953) in the paper by B. A. Trakhtenbrot [2]. Namely, B. A. Trakhtenbrot showed that the unsolvability of both problems of solva-

bility of narrow calculus, both the general problem and that on finite classes, follows directly from the circumstance that the set of all identically true formulas of narrow calculus of predicates is not separable recursively from the set of all formulas which are refuted in the finite one. His note [2] is indeed devoted to a proof of this non-separability. That problems of deductive incompleteness of a broad class of axiomatic theories of set are related with recursive non-separability of certain sets was observed by B. A. Trakhtenbrot already in the winter of 1949/50. Leaning on the existence of a pair of non-intersecting enumerable sets, not separable recursively, B. A. Trakhtenbrot in his dissertation (Kiev, 1950) proved, and furthermore assuming only the formal non-contradiction of the axiomatic theory of sets (of type [illegible] considered by him), the existence in it of formally unsolvable premises, which state the equivalence of certain conditions of finiteness of a set. (In the paper [11], which was published later, and which contains the exposition of these results of the dissertation, B. A. Trakhtenbrot changed the example he first used of a pair of recursively non-separable sets with an example from reference [2], which was discussed above). The incompleteness of set theory (in the sense of the existence in it of unsolvable, i.e., premises not provable and not refutable by its means, was thus proved to be the consequence of the recursive non-separability of certain sets.

Devoted to the general problem of the connection between incompleteness (and incompleteability) of formalized theory ("logical calculus") with the effective non-separability of certain sets was the work of V. A. Uspenskiy [5].

5. The work of V. A. Uspenskiy [5] was in answer to the question advanced by A. N. Kolmogorov concerning the general causes of the incompleteness of formalized theories which contain arithmetic. (This incompleteness (which can be treated both in the sense of the existence -- and, furthermore, effective in a definite sense -- of premises in the calculus under consideration which are not solved by its means, as well as in the sense of the existence in it of a formula that is interpretable as contentfully-true prediction concerning natural numbers and at

the same time not provable in this calculus), was proved, as is known by K. Goedel (1931) in his famous theorem, generalized later by Rosser (1936)[1.] As noted by V. A. Uspenskiy, the final formulation of the general definition of deductive calculus belongs to A. N. Kolmogorov. The class of calculi (formalized theories) which fall under this definition is quite broad. The only limitation imposed on their rule of derivation lies in that these rules must bear an algorithmic character (must be algorithms, which convert certain formulas (groups of formulas) into others). Also considered in the work are those calculi, the alphabet of which contains a definite symbol (for example, "$\neg$"), called the symbol of negation. In the $\mathfrak{A}$ calculus there is separated a class $\mathfrak{M}$ of formulas, closed with respect to operations of hanging the sign of negation, and such that there exists an algorithm, which recognizes whether any formula of the $\mathfrak{A}$ calculus belongs to $\mathfrak{M}$ or not.[2.] All the concepts introduced later on (non-contradiction, completeness, strengthening of the calculus, its non-completability, etc.) are considered as applied to a certain fixed set $\mathfrak{M}$. With the aid of the Goedel numbering, to each formula of $\mathfrak{A}$ calculus there corresponds a number -- the number of the formula. Principal attention is paid in this work to a class of calculi, in which for any formula $A \in \mathfrak{M}$ there follows from the derivability of A the derivability of $\neg\neg A$ and from the derivability $\neg\neg\neg A$ there follows the derivability of $\neg A$. Such calculi are called by V. A. Uspenskiy /5/ regular. (The constructive calculi, considered in Section 11, belong to the class of regular ones).

The class of formulas from $\mathfrak{M}$, which are derivable in the $\mathfrak{A}$ calculus, we denote $\mathfrak{M}(\mathfrak{A})$; the class of formulas (from the same $\mathfrak{M}$), the negation of which are derivable in $\mathfrak{A}$ (i.e., contained in $\mathfrak{M}$), we denote by $\mathfrak{M}(\mathfrak{A})$. We set in correspondence to these classes the

1. For bibliography, see, for example, the book by S. C. Kleene (Meta-Mathematics).
2. In the usual interpretation it is advisable to use for $\mathfrak{M}$ a certain set of "imaginable" formulas.

classes $\mathfrak{K}(\mathfrak{U})$ and $\mathfrak{L}(\mathfrak{U})$ of the numbers of formulas, which enter, (respectively) in $\mathfrak{B}(\mathfrak{U})$ and $\mathfrak{S}(\mathfrak{U})$.[1] The $\mathfrak{U}$ calculus is called non-contradictory if $\mathfrak{B}(\mathfrak{U}) \cap \mathfrak{S}(\mathfrak{U}) = \Lambda$; it is called complete if $\mathfrak{B}(\mathfrak{U}) \cup \mathfrak{S}(\mathfrak{U}) = \mathfrak{F}$. The calculus $\mathfrak{U}'$ is called the strengthening of the calculus $\mathfrak{U}$, if $\mathfrak{B}(\mathfrak{U}') \supset \mathfrak{B}(\mathfrak{U})$. A calculus is called incompletable if it does not admit of a complete and non-contradictory strengthening.

It is found that the non-separability of sets $\mathfrak{K}(\mathfrak{U})$ and $\mathfrak{L}(\mathfrak{U})$ is a necessary and sufficient condition for the non-completability of the regular calculus $\mathfrak{U}$.[2] (As applied to non-regular calculi, the non-separability of the sets $\mathfrak{K}(\mathfrak{U})$ and $\mathfrak{L}(\mathfrak{U})$ stops being, to be sure, a necessary condition of the non-completability of the calculus $\mathfrak{U}$, however, it remains in sufficient condition of the non-completability of an arbitrary calculus (V. A. Uspenskiy [5], theorems 2, 6).)

Moreover, by introducing the concept of effective non-separability and effective non-completability, analogous to what is done in the definition of a creative set, V. A. Uspenskiy proved that these theorems remain true also when the words "non-completability" and "non-separability" are replaced in them by the words "effective non-completability" and "effective non-separability." (By "effective non-separability" of two sets E_1 and E_2 one has in mind the systens of such a partially-recursive function $\gamma(n_1, n_2)$, which is defined for the numbers n_1 and n_2 of any enumerable sets H_1 and H_2, which separate E_1 and E_2, and attributes to them a natural number $\gamma(n_1, n_2)$, which belongs to neither H_1 of to H_2. If H_1 and H_2 are sets of numbers of formulas of which the first is derivable

---

1. On the rule of the derivation of Π calculus one imposes here an additional requirement (refined somewhat further by A. A. Markov), that to each of these there correspond a partially-recursive function, which converts the number of the formula, to which the given rule of derivation is applicable, to the number of the formula obtained from this rule. Under this condition the sets $\mathfrak{K}(\Pi)$ and $\mathfrak{L}(\Pi)$ are both enumerable.

2. The indication of the sufficiency of this condition belongs to A. N. Kolmogorov.

and the second -- refutable[1] in any strengthening P' of the P calculus, then it is clear that $\gamma(n_1, n_2)$ is the number of the formula which is not solvable in P', indicated effectively in this manner. The example, belonging to P. S. Novikov and B. A. Trakhtenbrot, of recursively non-separable non-intersecting denumerable sets are at the same time examples of effectively non-separable sets.)

If a non-contradictory regular P calculus is such that it has enough means to express and prove (contentfully) the statement that the number m belongs to the set E_1, where E_1 is one of two effectively non-separable denumerable sets E_1 and E_2, then it is easy to obtain from Uspenskiy's theorems that the P calculus is effectively non-completable,[1] i.e., that for any of each reinforcements one can indicate algorithmically a formula which is not solvable in this reinforcement. Since the deductive calculi, which describes arithmetic, are usually reinforcements of such a P calculus, this means that for these Goedel's theorem should hold. Other properties of formalized systems, conteining an arithmetic, were used by Goedel in proving his theorems (for example, the possibility of expressing the syntax of a system by means of the system itself) are found to be thus not essential conditions for its correctness. The essence of the matter lies indeed in the effective non-separability of sets of numbers of proved and refuted formulas of the system (or even of its part $\mathfrak{B}$). (The effective non-separability of these sets is found to be a sufficient condition of effective non-completability of an arbitrary calculus including also an irregular one (V. A. Uspenskiy [5], theorem 8).)

From among the other theorems and concepts, considered in the article by V. A. Uspenskiy [5], we note only the one pertaining to the connection between solva-

1. V. A. Uspenskiy [5], item 5) gives a method of effective construction under very natural conditions, from a pair of effectively non-separable sets E_1 and E_2, of such a calculus P_0, which is effectively non-completable and the reinforcements of which are all the general deductive calculi that describe arithmetic.

bility and essential non-solvability of a calculus with its non-completability. (According to Tarski, a calculus is essentially not solvable if it is non-contradictory and does not admit of a non-contradictory solvable reinforcement. As in other concepts introduced by him, V. A. Uspenskiy relativizes the concepts of solvability and essential non-solvability, assigning them to $\mathcal{B}$.) Indeed, it is found ([5], theorem 3) that a regular calculus is essentially not solvable when and only when it is non-contradictory and not completable. If the calculus is irregular, then this is generally speaking not true: there exist (non-contradictory) non-completable and at the same time solvable (as applied to the corresponding $\mathcal{B}$) calculi (V. A. Uspenskiy [5], theorem 7; the latter contains also a description of a certain class of such calculi).

As already noted above (Section 8, item 2), A. A. Muchnik gave an affirmative answer to the first two of the five questions raised by V. A. Uspenskiy [5]:

I. Do there exist non-separable sets which are not effectively non-separable?

II. Do there exist completable calculi, which are not effectively non-completable?

6. Developing further the ideas of V. A. Uspenskiy [5] and of J. Myhill, A. A. Muchnik[1] obtained several theorems on the properties of pairs of effectively non-separable sets and more generally such finite systems of enumerable sets,[2] which he called effectively multiply non-separable. In particular, it was found that for any pair of non-intersecting denumerable sets E_1, E_2 there exists a general-recursive function ψ, mutually uniquely mapping the natural series in itself, such that the sets of images $\psi(E_1)$ and $\psi(E_2)$ are effectively non-separable.

The circumstance that (theorem 2) all pairs of

1. A. A. Muchnik. Isomorphism of Systems of Denumerable Sets with Effective Properties. Transactions Moscow Mathematical Society 7 (1958), 407 -- 412 (reported on 17 December 1957).

2. This paper refers only to sets of natural numbers.

of effectively non-separable sets are isomorphic[1] to each other, was used to obtain further that effectively non-completable calculi are twice isomorphic between each other. (The deductive calculi L_1 and L_2 are called twice isomorphic if there exists an algorithm which placed in a mutually-unique correspondence all formulas of one system to all formulas of another system, and derivable formulas in L_1 correspond to derivable formulas in L_2, while refutable formulas in L_1 correspond to refutable formulas in L_2.)

(The remark of P. Bernays[2] concerning the formal systems of Myhill, which are "non-reducible to each other," pertain naturally also to twice-isomorphic calculi: any quite elementary system of arithmetic (without the axiom of complete induction) can prove to be twice isomorphic to a strong system of axiomatic theory of sets or theory of types with the assumption, naturally, of their non-contradiction.)

7. In connection with the Goedel theorem (on the incompleteness of formalized systems that describe arithmetic) the question arises naturally of whether it is possible to make the system become complete by adding to the formal system of certain non-finite, but at the same time sufficiently naturally and visualizable rules of deduction. The simplest of this type of rules is the rule of infinite induction, of which we spoke in section 8 (item 6). We dwelled there on the results of A. V. Kuznetsov [5] who proved that a (formalized) arithmetic with the rule of constructively infinite induction is already complete. More complicated formalized systems (which describe the classical mathematical analysis) with the ordinary rule of infinite induction ("the Carnap rule") but which admits the application of this rule not more than a transfinite ordinal number α times, was

1. Systems of sets $\{E_1, E_2, \dots, E_l\}$ and $\{F_1, F_2, \dots, F_l\}$ are called isomorphic if there exists a general recursive function ψ, mutually-uniquely mapping the natural series on itself, so that $x \in E_k \equiv \psi(x) \in F_k$ $(k = 1, 2, \dots, l)$.

2. P. Bernays, Journal of Symbolic Logic 22, (1957), 73 -- 76.

engaged by a student of P. S. Novikov -- B. Ya. Falevich.[1] The first chapter of the dissertation by. B. Ya. Falevich is devoted to an answer to the question of B. Rosser, raised as early as in 1937. Rosser succeeded in showing that if one adds to a logical-mathematical systme, which contains a set theory with simple theory of types and system of Peano axioms for the arithmetic of natural numbers, a deduction rule which permits the use of the "Carnap rule" α times, where α is a transfinite ordinal number less that ω^2, then both Goedel theorems (both regarding the incompleteness of the calculus as well as the non-provability of the non-contradiction of the calculus by its own means) retain their force. Why, however, must α be less than ω^2? Could it be that for larger α the Goedel theorems lose their force? In his dissertation, B. Ya. Falevich showed (to be sure, in application to certain different systems, which describe the classical mathematical analysis), that ω^2 enters here not all because under very broad conditions (which are satisfied by broad classes of constructive transfinite α) for systems admitting the application of the Carnap rule α times, both Goedel theorems remain in force.

Another question which engaged the attention of B. Ya. Falevich in his dissertation pertains to the question of the relation between classical and constructive formalized systems of mathematical analysis. Namely, to the question whether there exists such a calculus (constructive in a definite sense), which formalizes the mathematical analysis, the non-contradiction of which would be proved and which one could "submerge" (exactly as the classical arithmetic of rational number is submerged in the constructive arithmetic of the same numbers) the classical mathematical analysis, proving thereby its non-contradiction. Naturally, B. Ya. Falevich did not succeed in finding an answer to this question, the difficulty of which is evident from the entire history of modern mathematical logic. He did show, however, that even in one of

1. B. Ya. Falevich, Incompleteness Theorems in Systems with the Carnap Rule and Their Applications. Dissertation, Moscow, 1956.

-1-

the strongest of the heretofore-developed systems of constructive mathematical analysis (that of W. Ackermann, the proof of non-contradiction of which has been fully demonstrated by the author) there exists no "correct" model for the system S_0 of classical analysis (without Carnap's rule), considered by B. Ya. Falevich ("correct," i.e., satisfying definite requirements, which, incidentally, are quite natural). And a constructive analysis, in the sense of Ackermann, is thus incapable of duplicating the contents of classical mathematical analysis.

8. The second Goedel theorem, namely concerning the provability of non-contradiction of calculus by means of this calculus itself, engaged the attention of not only B. Ya. Falevich, but also of A. S. Yesenin-Vol'pin. His results, reported at the Third All-Union Mathematical Congress (see also [4]) consist of the following.

I) An example is constructed of a formula that expresses the non-contradiction of a system which at the same time is no less provable in this system. This simple example shows that formulations of the second Goedel theorem the concept of "numerical expressibility" of the predicate,[1] which is sufficient for Goedel's first theorem, is insufficient for the formulation of the second theorem.

II) It was established that it is impossible to prove in the ω-non-contradictory calculus, satisfying conditions I of Goedel's theorem, a formula that states that "if this calculus is non-contradictory, then it is also ω-non-contradictory," and constructive in the way required in Goedel's second theorem.

9. The principal problem connected with "logical calculi" formalized theories) pertains to their relationship to contentful models, or, in other words, interpretations. Essentially, a formal system is necessary precisely in order to help clarify (and refine) the contents of the theory formalized with its aid. The most important value of theorems on incompleteness and on incompletability is due to the fact that they clarify the limits of the capabilities of such a formalization. On the other hand, it must be noted that for a broad class of incomplete (and

1. S. C. Kleene, Introduction into Meta-Mathematics, Moscow, Foreign Literature Press, 1947, Section 41.

incompletable) regular calculi, considered by V. A. Uspenskiy [5] it is possible to indicate algorithmically in them an unsolvable formula Φ such that $\neg\Phi$ is contentfully true, but not derivable. The incompleteness of the calculus therefore does by no means denote the impossibility of establishing a contentful truth (or falsity) of a formula which is not solvable in this calculus.

The circle of problems connected with the relation between formal theories and their contentful model pertains to the field of semantics, the scope of which is getting more and more segregated into a special part of mathematical (and general) logic. The theory of models is being developed, however, also in the form of a certain mathematical theory, closely related with modern algebra. Engaging in our country with such a theory was A. I. Mal'tsev, on whose work we shall now dwell briefly. We shall note as a preliminary only that these works have the character of classical mathematical researches, no longer connected with the theory of algorithms.

The ordered system $\mathfrak{M} = (A, R_1, R_2, \ldots, R_k, \ldots)$, where A is a non-empty set, and R_k is a n_k-place predicate on A, is called by A. A. Mal'tsev a model. The type of the model $\mathfrak{M}$ is called a row of natural numbers $(n_1, n_2, \ldots, n_k, \ldots)$, corresponding to the number of argument places in the predicates R_k of the model $\mathfrak{M}$. From now on we shall understand by class of models the class of single-type models, which contain together with any of their term all the isomorphic ones. We fix a certain type $(n_1, n_2, \ldots, n_k, \ldots)$ and consider the class Q of all models of this type. We construct a formal system corresponding to this class. For this we take a sequence of object symbols $(a_1, a_2, a_3, \ldots)$, a sequence of symbols for the variables $(x_1, x_2, x_3, \ldots)$ and a sequence of predicate symbols $(P_1, P_2, P_3, \ldots, P_k, \ldots)$, where P_k is a symbol of the n_k-place predicate. We call an axiom a closed formula of the narrow calculus of predicates, compiled of these symbols. For an arbitrary fixed axiom and an arbitrary fixed model from Q, the following question is raised in a natural manner: Is this axiom fulfilled in this model or not? One can specify a certain set of axioms, having certain properties and ask: how is it possible to characterize the class of all these and only these models (from Q), in which all these formulas are

satisfied? The question can also be stated inversely. Separate in class Q of all models of a given type a certain subclass of models, having certain special algebraic properties and ask: do we axiomatize this subclass or — a narrower question — do we axiomatize this subclass with the aid of axioms of a certain concrete type? For example, a sufficient condition of the representability of a certain model of a certain class K by a model in the form of a sub-straight product of K — non-expandable K models was found by A. I. Mal'tsev in [45] in the following form: the class K should be axiomatizable with the aid of axioms of two types:

$$(Q_1 x_1) \ldots (Q_m x_m)\, \mathfrak{A}(x_1, x_2, \ldots, x_n), \tag{13}$$

where Q_i is any of the quantors, and $\mathfrak{A}(x_1, x_2, \ldots, x_n)$ is a formula made up of expressions of the form $P_t(x_{i_1}, x_{i_2}, \ldots, x_{i_{n_t}})$ only with the aid of the operations $\&, \vee$;

$$\forall x_1 \ldots \forall x_m \mathfrak{B}(x_1, x_2, \ldots, x_m), \tag{14}$$

where $\mathfrak{B}(x_1, \ldots, x_m)$ is a formula compiled of the expressions of type $P_t(x_{i_1}, x_{i_2}, \ldots, x_{i_{n_t}})$ with the aid of the operations $\&, \vee, \bar{}$.

In [46] A. I. Mal'tsev proved the following theorem: in order that a finite axiomatizable class of models be closed relative to taking of homomorphism, it is necessary and sufficient that it be axiomatizable with the aid of axioms of the type (13).

In [52] A. I. Mal'tsev introduced the concept of pseudo-axiomatizability such that any axiomatized class is found to be pseudo-axiomatizable, but not vice versa. In the same reference [52] A. I. Mal'tsev found a sufficient condition for having a pseudo-axiomatized class become axiomatizable. It was found that when this sufficient condition is satisfied, a pseudo-axiomatizable class is axiomatizable only with the aid of axioms which are written in the Skolem normal form.

An ordered system $\mathfrak{A} = \langle A, f_1, \ldots, f_t, \ldots \rangle$, where A is a non-empty set and f_t is a n_t-place operation, is called an algebra. For any n_t-place operation f_t it is possible to define in a natural manner $(n_t + 1)$-place predicate R_t, by putting $R_t(a_1, a_2, \ldots, a_{n_t}, b) \equiv (f_t(a_1, \ldots, a_{n_t}) = b)$. The algebra becomes a model. The inverse operation is not always

possible. From the model it is obviously possible to make an algebra if and only if for any basic predicate R_i the axiom of existence and single-valuedness is satisfied

$$\forall x_1 \forall x_2 \ldots \forall x_{n_i-1} \exists y R_i(x_1, x_2, \ldots, x_{n_i-1}, y) \, \&,$$
$$\forall x_1 \ldots \forall x_{n_i-1} \forall u \forall v [R_i(x_1, \ldots, x_{n_i-1}, u) \, \& \, R_i(x_1, \ldots, x_{n_i-1}, v) \rightarrow u = v]. \tag{15}$$

For classes of algebras the questions also arise of axiomatizability with the aid of axioms of one form or another. In /44/ A. I. Mal'tsev found two algebraic criteria of axiomatizability with the aid of conditional identities, i.e., formulas of the type:

$$\forall x_1 \ldots \forall x_m [\varphi_1 = \psi_1 \, \& \ldots \& \, \varphi_n = \psi_n \rightarrow \varphi_{n+1} = \psi_{n+1}], \tag{16}$$

where $\varphi_1, \psi_1, \ldots, \varphi_{n+1}, \psi_{n+1}$ are polynomials in the variables $x_1, \ldots, x_m$. Any non-closed formula of the narrow calculus of predicates constructed in a form of system corresponding to a certain class of algebras, can be considered as a derivative predicate on any of the algebras of this class. But by far not any such formula will represent an operation in the sense of axiom (15). In /51/ A. I. Mal'tsev established a general form of operations, obtainable with the aid of formulas of narrow calculus of predicates, for the case of algebras characterizable with the aid of axioms of type (14). In the same paper /51/ is given an abstract of characteristic of predicates, representable by conjunctions of formulas of narrow calculus of type (14).

10. From the point of view of algebraic applications, of great interest is the Goedel theorem concerning the completeness of the narrow calculus of predicates. The point is that with the aid of this formula it is possible to prove many theorems concerning theorems of algebra, making it possible to obtain immediately entire series of algebraic theorems. A source of such theorems concerning theorems is, above all, a general theorem obtained by A. I. Mal'tsev /7/ in 1941 with the aid of the results of his work of 1936 /1/, devoted to a generalization of the Goedel theorem. The theorem of A. I. Mal'tsev reads as follows: For an infinite system of formulas of narrow calculus of predicates, admitting the relation

of identity and an arbitrary set of symbols for individual objects and predicates, to be compatible it is necessary and sufficient that each finite subsystem of the given system be compatible.

A considerable number of algebraic local theorems can be derived from this principal local theorem. In the same reference [7] this formula was applied to the solution of certain problems in group theory which at that time were not yet solved.

This theorem was also used by A. I. Mal'tsev (in [32]) and by A. A. Vinogradov (in [2, 3]) in the theory of ordered groups.

To prove concrete local theorems of algebra it is usually necessary to introduce auxiliary constructions. In [32] A. I. Mal'tsev derived from the basic local theorem three more particular local theorems, which do not require these auxiliary constructions. In the same reference [32] one of these theorems was applied to the theory of ordered groups.

Another general theorem of the same kind as the basic local theorem of A. I. Mal'tsev was obtained by Yu. A. Shikhanovich [1] through a generalization of the scheme of proof of one of the theorems on theorems of A. Robinson.[1] The theorem, proved by Yu. A. Shikhanovich, consisted of the following: let us take an arbitrary sequence of axioms

$$H_0, H_1, \ldots, H_n, \ldots \tag{17}$$

e inductively define with the aid of (17) a new sequence

$$X_1 = H_0,$$

$$X_n = [X_{n-1} \& H_{n-1}] \quad (n = 2, 3, 4, \ldots).$$

e denote the class of axioms $\{X_1, X_2, \ldots, X_n, \ldots\}$ by P. Then:

Theorem 1. If a certain definitely-axiomatizable class of models, definable by the axiom Y, contains any model, then it contains also any X_{n_0} model, where n_0 is constant that depends on Y.

Theorem 2. If there exists a model for X_n, but not for P, for a value of n as large as desired, then the class is not axiomatized with the aid of a finite number of

[1] A. Robinson. On the Meta-Mathematics of Algebra. 1951.

axioms.

With the aid of this theorem Yu. A. Shikhanovich [1] obtained several new examples of theorems analogous to the Robinson theorem.

13. Algebra of Logic and Its Generalizations[1].

1. The simplest part of mathematical logic, and furthermore the earliest among all other of its sections, is the algebra of logic. This is precisely why it appeared for a long time that no interesting unsolved problems remained in it. However, in the last decade interesting and difficult problems were exhibited in this field, and a number of new results and papers began to increase rapidly, perhaps faster than in many other branches of mathematical logic. This is connected above all with the fact that with the rigorous growth of automatization and telemechanization of industry, the role of those problems which the theory of relay-contact circuits has raised increased, and this theory has been using algebraic logic for a long time. Many of the problems in this theory are found to be impossible to solve by previously known methods and required the development of new ones, including also methods in algebraic logic. Furthermore, the appearance of ever more and more new types of electric circuits (electronic, magnetic, etc.) and the complication of those previously known, requiring an adequate mathematical representation of functions and their elements, has given rise to the necessity of engaging also in such cases, in which the initial functions are not ordinary negation, disjunction, and conjunction, but various other functions. This in turn has made it necessary to study various multiple-valued generalizations of ordinary algebraic logic.

Work in these latter directions began in our country already in 1950 -- 1951. The initial points of this work were the lectures by P. S. Novikov at the Moscow University in the fall of 1950, in which he formulated many

1. Written by A. V. Kuznetsov, edited by A.A. Yanovskaya.

problems in the field of algebraic logic and indicated certain approaches to their solution, as well as the diploma work of Yu. I. Yanov (spring 1951, guided by A. A. Yanovskaya), connected with his analysis of the results contained in the article by Rosser and Turquette "Many-Valued Logics."

One of the questions raised by P. S. Novikov was as follows: what are the necessary and sufficient conditions in order that from among the functions $\Phi_1, \Phi_2, \ldots, \Phi_n$ of algebraic logic[1] one could obtain any other function of algebraic logic by superposition (i.e., by insertion of functions in a function or by insertion of variables in a function). He also showed in his lectures that the necessary conditions for the latter are, for example, the following: 1) The equality $\Phi_i(t,\ldots,t) = t$ is not true for all $i=1,2,\ldots,n$; 2) the equality $\Phi_i(f,\ldots,f)= f$ is not true for all $i=1, 2,\ldots,n$; 3) at least one of the functions $\Phi_i (i=1,2,\ldots,n)$ is self-dual, i.e., it does not satisfy the identity

$$\overline{\Phi_i(x_1,\ldots,x_s)} = \Phi_i(\bar{x}_1,\ldots,\bar{x}_s).$$

Soon later a graduate student of P. S. Novikov, S. V. Yablonskiy solved this question for the case $n = 1$, showing that in this case the conjunction of the three necessary conditions, indicated by P. S. Novikov, is also a sufficient condition.[2] In the spring of 1951 S. V. Yablonskiy solved the problem completely, proving the following theorem: In order to be able to represent with the aid of the functions $\Phi_1,\ldots,\Phi_n$ of algebraic logic any function of algebraic logic, it is necessary and sufficient that for this system of functions the conditions 1) -- 3) of P. S. Novikov be satisfied, and in addition the following additional two: 4) at least one of the functions $\Phi_i (i=1,\ldots,n)$ is not representable through the functions $\bar{x}$, $x \equiv y$; 5) at least one of the functions $(i=1,2,\ldots,n)$ is different from the constants

1. That is, such functions, the arguments of which assume values t (truth) and f (false) and the values of which (functions) can also be these t and f.

2. Reported on the Seminar on Mathematical Logic at the Moscow State University at the end of November 1950.

t, f and is not representable in terms of the functions $x\&y$ and $x \vee y$.[1] By representability of a function Φ in terms of functions $\Phi_1, \ldots, \Phi_n$ we understand here the possibility of obtaining the function Φ from the functions $\Phi_1, \ldots, \Phi_n$ by superposition. If, as is customary, the property of the set of functions $\{\Phi_1, \ldots, \Phi_n\}$ is expressed by the fact that one can represent through them all the remaining functions, and if this property is called functional completeness and if it is noted that by identifying (conditionally) the truth (t) with one and falseness (f) with zero, we find that the representability of the function through $\bar{x}$ and $x\&y$ is equivalent to its linearity according to I. I. Zhegalkin [2], i.e., in the sense of a ring of residues mod 2, and the representability of the function in terms of 0, 1, xy[2] and $x \vee y$ is equivalent to its monotonocity[3] (if one assumes, as usual, that $0<1$), then the theorem assumes the following form: for a functional completeness of a set A of functions of algebraic logic it is necessary and sufficient that A not be included in (or equal to) either of the following sets: 1) The set of such Φ, that $\Phi(0, \ldots, 0)=0$; 2) the set of such Φ, that $\Phi(1, \ldots, 1)=1$; 3) the set of self-dual functions; 4) the set of linear functions; 5) the set of monotonic functions. Later on A. V. Kuznetsov showed that these five sets cannot be replaced in the theorem by any others and have the theorem remain true. The latter is connected with the fact[4] that these five sets and only these five are the so-called precomplete closed sets of functions of algebraic logic. Here the closure (functional) of a set B denotes that any function represented through functions from B also belongs

1. This formulation of the theorem is close to that given by S. V. Yablonskiy in his lecture at the Seminar on Mathematical Logic at the Moscow State University on 17 March 1951. In 1952 S. V. Yablonskiy published this theorem in [3].
2. After identifying t with 1, and f with 0, the conjunction x & y becomes a multiplication xy mod 2.
3. More accurately, monotonic non-ciminishability, or, in other words, isotopy.
4. See theorem 3) of item 4.

to B. A set B is called, according to A. V. Kuznetsov[1], pre-complete when B is not functionally complete, but is such that any set containing all the functions from B and at least one more, not representable in terms of these, already becomes complete.

In a survey paper delivered to the session of the Moscow Mathematical Society in the fall of 1951, A. N. Kolmogorov considered the particular case, when in addition to substitutions of certain functions in other or certain variables in a function, one admits also the substitution of constants into a function during the superposition. As a consequence of the foregoing theorem he obtained the following theorem: in order for a set A of functions of algebraic logic to be complete relative to such superpositions,[2] it is necessary and sufficient that there be contained in A at least one nonlinear function and at least one non-monotonic one. It was soon noted that this theorem, proved by S. V. Yablonskiy, is contained (although in a somewhat different form) in the book by Post, which was published as early as 1941, but which was unknown to us. True, the proof of S. V. Yablonskiy is simpler and shorter than that of Post, who obtained this theorem as a corollary of a survey of all the closed sets of functions of algebraic logic. At the beginning of 1953 A. V. Kuznetsov obtained an even shorter and simpler proof of this theorem,[3] which serves as the basis for the proof of the theorem of A. N. Kolmogorov given above as a lemma, and not as a corollary.

2. The theorem itself (that of Post-Yablonskiy) was somewhat generalized by A. V. Kuznetsov to include the case of many-valued logic, which in its functional construction (S. V. Yablonskiy [7]) is taken to mean a theory of

---

1. A. V. Kuznetsov, Abstract of the article by S. V. Yablonskiy "On Functional Completeness in Three-Valued Calculus" Referat Zhur Matematika, No. 1, 1955.
2. Weakly complete, according to the terminology of the paper by A. V. Kuznetsov [3].
4. This proof was published in the works of the Mathematical Institute imeni Steklov, 51 (1958), 18 -- 20.

functions which are defined in a set $E^k = \{0, 1, 2, \ldots, k-1\}$, in that sense, that the arguments of the functions assume all values from E^k and the values of the function itself also belong to E^k. It was found that also for the case of a function of such a k-valued logic the same holds: for the set A of (any) functions of k-valued logic to be complete it is necessary and sufficient that A not be included (or equal to) in either case of the pre-complete closed sets of functions of k-valued logic. It remained only to find all the pre-complete closed sets. In 1953 S. V. Yablonskiy [6] obtained a list of all the pre-complete closed sets for the case k = 3 (their number was found to be 18), and formulated this result[1] in the form of a theorem,[2] analogous to the Post-Yablonskiy theorem. A simpler and compact form was obtained for this list by A. V. Kuznetsov, who leaned on the concept, which he introduced already in the spring of 1951, of retention of the predicate. Concerning the function $\Phi(x_1, \ldots, x_n)$ one says that it retains the predicate $P(x_1, \ldots, x_s)$, if the formula

$$P(x_{11}, x_{12}, \ldots, x_{1s}) \& P(x_{21}, x_{22}, \ldots, x_{2s}) \& \ldots \& P(x_{n1}, x_{n2}, \ldots, x_{ns}) \supset$$
$$\supset P(\Phi(x_{11}, x_{21}, \ldots, x_{n1}), \Phi(x_{12}, x_{22}, \ldots, x_{n2}), \ldots, \Phi(x_{1s}, x_{2s}, \ldots, x_{ns}))$$

is true for all values of the variables x_{ij} $(i = 1, 2, \ldots, n;$ $j = 1, 2, \ldots, s)$. The set of all functions which retain the predicate P is called the class of retention of the predicate P. It is obvious that for any predicate the class of its retention is a closed set. Those five sets, which enter into the Post-Yablonskiy theorem, are classes of retention of the following predicates (defined on

$$\{0, 1\}): \ x = 0, \ x = 1, \ x \neq y, \ x + y = z + u, \ x \leqslant y$$

(respectively), where the sum is taken to be a sum

1. S. V. Yablonskiy. Problems of Functional Completeness in k-valued Calculus. Abstract of dissertation, Moscow, 1953.
2. A complete proof of it is found in the dissertation of S. V. Yablonskiy, published in Trudy matem in-ta im. Steklova (Works of the Mathematical Institute imeni Steklov), 51, (1958).

mod 2[1] and the 18 classes from the theorem of Yablonskiy for three-valued logic are classes of retention of the following 18 predicates (defined on $\{0, 1, 2\}$):

$$x = i, \quad x \neq i, \quad x + 1 = y, \quad x + y = z + u, \quad x + i \leqslant y + i,$$

$$(x = i) \equiv (y = i), \quad x = y \vee x = i \vee y = i, \quad x = y \vee x = z \vee y = z,$$

where i is a parameter that assume the values 0, 1, and 2, while the sum is taken in mod 3. Already in 1951 A. V. Kuznetsov proved that any prefilled closed set of functions of k-valued logic is a class of retention of a certain predicate, which depends (for $k \geqslant 3$) on not more than k arguments. From this he obtained the upper estimate for the number μ_k of pre-complete closed sets: $\mu_k < 2^{k^k}$. However, attempts to find an exact estimate of the number and, furthermore, to obtain their total list for a general k-valued case entail great difficulties. A. V. Kuznetsov and S. V. Yablonskiy have constructed a series of families of pre-complete closed sets, which generalize the preceding examples to a k-valued case. These included the classes of retention of the following predicates: $x \in E$, where E is not empty and differs from $\{0, 1, \ldots, k-1\}$; $\varphi(x) = y$, where φ is such a function, that the substitution

$$\begin{pmatrix} 0 & 1 & 2 & \ldots & k-1 \\ \varphi(0) & \varphi(1) & \varphi(2) & \ldots & \varphi(k-1) \end{pmatrix}$$

is such that all those (independent) cycles, into which it breaks up, have the same length, equal to a certain simple number; any predicate of partial order[2] P(x,y), satisfying the condition

$$(Ex)(y)P(x, y) \,\&\, (Ey)(x)P(x, y);$$

any predicate of the equivalence type[3] which differs from x = y or from the identically true one. More complicated examples were also constructed. However, the number of these classes did not increase very rapidly with increasing k. And only in 1955 did A. V. Kuznetsov succeed in pointing the way towards proving that $\mu_k > 2^{2^k(1-\varepsilon)}$, where ε tends

1. Reported to the Seminar on Mathematical Logic at the Moscow State University by A. V. Kuznetsov on 26 May 1951.
2. That is, reflexive, transitive, and anti-symmetrical.
3. That is, reflexive, symmetrical and transitive.

to 0 with the increasing k.

Searches for a list of pre-complete closed classes and an estimate of the number are important in connection with the problem of perfecting algorithms for the problem of recognition of completeness (of finite) sets of functions of k-valued logic. The very existence of such an algorithm, a single one for all k, was proved already in 1951 by A. V. Kuznetsov,[1] although the initial version of this algorithm was little suitable in practice, owing to the cumbersomeness of the derivation. As to the problem of recognizing the pre-completion of sets of functions, it is still unknown whether there exists a single algorithm for all k. A. V. Kuznetsov [3] proved in 1956 the existence of an algorithm corresponding to this problem for each fixed k, and the proof is non-constructive.

Also investigated was the question of what number σ_k is maximal for all possible cardinalities of independent sets of functions of k-valued logic. Here the set A is called independent if no function $\Phi \in A$ can be represented through other functions from A. It is obvious (A. V. Kuznetsov) that $\sigma_k \leq p_k$. S. V. Yablonskiy showed ([3] and [7]) that $\sigma_2 = 4$ (an example of a complete independent set of four functions: $\{0, 1, xy, x+y+z\}$). $6 \leq \sigma_3 \leq 7$ (example: $\{0, 1, 2, \min(x, y), \min(x+1, y+1)+2, \min(x+2, y+2)+1\}$, where the addition is in mod 3).

The difficulties connected with the absence of sufficiently convenient general criteria of completeness of sets of functions, raise an interest also in various particular problems, connected with an examination of sets of a definite type. These problems above deal with (functional) completeness of (universal) algebras. An algebra $\mathfrak{A}$ with operations $\Phi_1, \ldots, \Phi_n$ is called complete (or weakly complete) if the set $\{\Phi_1, \ldots, \Phi_n\}$ is complete (or, respectively, weakly complete)[2] as a set of functions.

---

1. Reported on the Seminar on Mathematical Logic at the Moscow State University, 26 May 1951. Mentioned by S. V. Yablonskiy [6].
2. In the sense of A. V. Kuznetsov [3]; see also footnote 3 on p. 104 [of source].

It is easy to note that no groups, rings, or lattices (which have more than one element) can be complete. However, as noted already in the spring of 1951 by A. V. Kuznetsov (who generalized the well known theorem of I. I. Zhegalkin concerning the completeness of the set $\{1, xy, x+y \pmod 2\}$), a ring of residues of integers in mod k is weakly complete when and only when k is a simple number. In 1955 A. V. Kuznetsov proved the following generalization of this theorem: for weak completeness of a ring it is necessary and sufficient that it be finite, simple, with non-zero multiplication.[1] He then proved that for weak completeness of a group it is necessary and sufficient that it be finite, simple, and non-Abelian. Both these theorems were found to be, as soon noted by A. V. Kuznetsov, particular cases of the thorem for quasi-ring (in the sense of A. I. Mal'tsev), analogous to the first of these. In 1956 A. V. Kuznetsov noted that any finite structure after adding to the number of its operation the operation $x \sim y$ ($x \sim y$ is equal to 1 (unit of the lattice) for x = y and is equal to 0 (null of the lattice) for $x \neq y$) becomes a weakly-complete algebra.[2]

4. In addition to the usual completeness and weak completeness there were investigated also other types of completeness (in the more general meaning of the word). The general statement of the problem of completeness was given in the lectures of A. V. Kuznetsov on algebraic logic, delivered by him at the Moscow State University in the fall of 1957. He laid the groundwork for a general concept of operations of closure for subsets of a certain fixed set M, and with this the operation $[A]$, defined for all $A \subseteq M$, is called, as usual,[3] the closure operation, if it has the following properties:

1°. $A \subseteq [A] \subseteq M$ (exteriorness, or extensiveness).
2°. $[[A]] = [A]$ (idempotency).
3°. if $A \subseteq B$, then $[A] \subseteq [B]$ (isotonicity).

---

1. That is, not for all elements a and b of the ring ab = 0.
2. Reported at the Seminar on Algebraic Logic at the Moscow State University in December 1956.
3. G. Birkhoff, Lattice Theory, Moscow, Foreign Literature Press, 1952.

The set A is called closed if $[A] = A$. A is called complete, if $[A] = M$. A is called pre-complete, if it is not complete, but for any element $\varphi \in M \setminus A$ the set $A \cup \{\varphi\}$ is already complete. The aggregate $\mathfrak{A}$ (of certain) subsets of the set M is called criterial if for any $A \subseteq M$ the following holds: A is complete when and only when there does not exist for it such a B, that $A \subseteq B \in \mathfrak{A}$. The aggregate $\mathfrak{A}$ is called completely criterial if it is criterial, but no regular part of it is criterial. All these concepts are defined with respect to given sets M and operations of closure of its subset. All these concepts are defined with respect to subsets relative to the set M and its closure operations.

From these definitions one naturally obtains (A. V. Kuznetsov) the following theorems:

1) The aggregate of all incomplete closed sets is criterial.

2) If there exists a finite criterial aggregate $\mathfrak{A}$, then there exists also a finite fully criterial aggregate $\mathfrak{B} \subseteq \mathfrak{A}$.

3) If there exists a fully criterial aggregate, then it is unique and coincides with the aggregate of all pre-complete closed sets.

From among the less general theorems proved by A. V. Kuznetsov, let us mention here the following two.

a) let M be a certain closed (in the sense of items 1 and 2) set of functions of k-valued logic, and the closure $[A]$ (where $A \subseteq M$) is a set of all functions from M, represented through the function from A; let there exist a complete set B, not containing the functions which depend on more than n arguments; then the aggregate of all classes of retention of predicates different from M, defined on $\{0, 1, 2, \ldots, k-1\}$ and which depend on not more than k^n arguments, as a finite criterial aggregate.

b) Let M be a set of all functions of k-valued logic, and let the closure $[A]$ $(A \subseteq M)$ be the set of all such functions $\Phi \in M$, that (for Φ) there exist such functions $\Phi_1, \ldots, \Phi_n \in M$ and such a system of functional equations, which uniquely defines a (n + 1) term collection of functions $\Phi : \Phi_1, \ldots, \Phi_n$ and, in addition to the signs of these defined functions and functions ...

belonging to A, it does not contain any other signs of functions;[1] then for completeness of the arbitrary set $B \subseteq M$ it is necessary and sufficient that the algebra with the element 0, 1, 2, ..., k-1 and operations belonging (as functions) to B have no non-trivial automorphisms.[2]

Another interesting example occurs when M is the set of all the functions of (ordinary) algebraic logic, and the closure $[A]$ is a set of all those functions $\Phi \in M$, which can be obtained from the functions belonging to A by inserting the variables into the function. It is found that the set of all symmetric functions[3] is found to be complete even in this sense. Indeed, G. N. Povarov calls [11] the function $\Phi(x_1, \ldots, x_n)$ quasi-symmetrical with respect to weighting $e_1, e_2, \ldots, e_n$ (weights ascribed to the variables), if there exists such a symmetrical function S that

$$\Phi(x_1, \ldots, x_n) = S(\underbrace{x_1, \ldots, x_1}_{e_1}, \underbrace{x_2, \ldots, x_2}_{e_2}, \ldots, \underbrace{x_n, \ldots, x_n}_{e_n}).$$

As noted by V. N. Roginskiy (see G. N. Povarov [11]), any function of algebraic logic $\Phi(x_1, \ldots, x_n)$ is quasi-symmetrical with respect to weighting $1, 2, 4, \ldots, 2^{n-1}$.

Another case is considered by S. V. Yablonskiy in [9] and [11]: M is the same as in the preceding example, and the closure $[A]$ is a set of all those functions, which are constant or are obtained from functions belonging to A by substitution of constants and substitution of variables, when equal variables cannot be substituted in the place of different ones. Sets closed in this sense are called by S. V. Yablonskiy invariant. He gives the following examples of invariant sets: the set of all linear functions, the set of all symmetric functions, the

1. Compare with theorems III and IV of item 5 of Section 8 of the present article.
2. We note that non-trivial automorphism is the retention by all the operations of the algebra of a certain predicate of the form $\varphi(x) = y$, where $\varphi(x)$ is a mutually-single-valued function different from x.
3. A function is called symmetrical, if it does not change under any rearrangement of its arguments.

set of all monotonic functions, the set of all functions that depend on not more than n variables. He then proves that the cardinality of the aggregate of all invariant sets of functions (if they cannot have arguments different from $x_1, x_2, \ldots, x_i, \ldots$) equals the cardinality of the continuum; the same holds also for the aggregate of only those invariant sets Q, for which $\lim\limits_{n\to\infty} \sqrt[2^n]{P_Q(n)} = 1$. Here $P_Q(n)$ is the number of functions from Q which depend on $x_1, \ldots, x_n$. S. V. Yablonskiy shows that, with increasing n, for any invariant Q $\sqrt[2^n]{P_Q(n)}$ tends without increasing to a limit that lies in the segment $[1, 2]$, and if $Q \neq M$, then $\lim\limits_{n\to\infty} \frac{P_Q(n)}{2^{2^n}} = 0$.

5. Certain theorems, which hold for k-valued logics, are carried over also to so-called infinite-valued logics. Thus, let us take the case when M is a set of all functions, which are defined in the set E (finite or infinite); in that sense, that the arguments assume all values from E and the values of functions also belong to E, and the closure $[A]$ is a set of all functions represented through functions from A. As proved already in 1951 by A. V. Kuznetsov, no matter what E may be, there exists such a set of functions $\Phi \in M$, which is complete, but in which all the functions with the exception of one depend on a single argument. (In the general case in the proof of this one uses the selection axiom, but it is not required when E is fully ordered, or, for example, continual).[1] The class of retention of predicate $x \in E'$, where $E' \subset E$, and is not empty, is also a pre-complete closed set for all such E' and E. However, in the case of an infinite E there no longer exists any finite complete set, or any finite criterial aggregate, this being connected with the fact that M is not denumerable.

It is therefore interesting to consider certain intermediate cases between k-valued logic and such infinite-valued ones. One family of such cases was the sub-

---

1. Reported at the Seminar on Mathematical Logic at the Moscow State University, 26 May 1951.

ject of a paper by S. V. Yablonskiy at the Seminar on Mathematical Logic, held at the Moscow State University on 23 October 1957.

S. V. Yablonskiy considers an example, when the foregoing set E is a natural series, and seeks such closed sets $P \subset M$, which satisfy the following conditions: 1) P is denumerable; 2) for any natural k there exists such a $P_k \subset P$, which is homomorphically mapped on the set of all functions of k-valued logic. Such P he called limiting logics. (Homomorphic mapping of one set of functions on another is called here by S. V. Yablonskiy such a mapping, under which sets of arguments of functions from these two sets are placed in mutually-unique correspondence, the image of the function is a function of the corresponding arguments, and the image of the result of the superposition is a result of the corresponding superposition of the images. Homomorphism in both directions, as usual, is called isomorphism). S. V. Yablonskiy gives simple examples of limiting logics and then constructs a continual family of pairwise non-isomorphic limiting logics.

6. Many other problems of algebraic logic, in which we are engaged here, are more directly connected with the theory of relay-contact circuits or other devices of relay action and usually arise under the influence of its needs. Being unable to discuss in this article all the investigations in this field, we shall touch only upon certain of these the results of which pertain to the algebraic logic itself, and not only to its applications.

Among investigations of this kind are those devoted to problems of functional separability and so-called unrepeated superpositions. Investigations of these problems began in our country in 1951 and were caused by an evaluation of various difficulties, which arise in the study of bridge-type[1] contact circuits. The starting point was the remark, made at the Seminar on Mathematical Logic of the Moscow State University by P. S. Novikov, on the

1. That is to say, not obtained from trivial ones by merely parallel and series connections, adequately represented by operations of disjunction and conjunction.

absence of the mathematical proof of the fact, verified by many ears experience and stated many times by V. I. Shestakov in his lectures, that even for the simplest bridge circuits, the corresponding function $M(a,b,c,d,e)=ad\vee ace\vee bcd\vee be$ cannot be represented through functions of two arguments in such a way that in the expression serving for its representation not one of the letters would be encountered more than once.

The impossibility of such a representation, called by A. V. Kuznetsov the unrepeated superposition, was soon proved by the latter. With this, A. V. Kuznetsov proved more for the function M(a,b,c,d,e): it cannot be represented by an unrepeated superposition in terms of functions of a smaller number of arguments than the function itself. The function with this property are called (functionally) non-separable. Then A. V. Kuznetsov proved a more general theorem on non-separability of any function from a certain sufficiently broad class of functions, connected with bridge circuits.[1] The results of A. V. Kuznetsov are based on the following lemma: in order for a subset $\{x_{i_1},\dots,x_{i_m}\}$ of a set of arguments of functions of algebraic logic $\Phi(x_1,\dots,x_n)$ to be selectable in the sense of the existence of such functions φ and χ, that

$$\Phi(x_1,\dots,x_n)=\varphi(\chi(x_{i_1},\dots,x_{i_m}),x_{i_{m+1}},\dots,x_{i_n}),$$

where all indices $i_1,\dots,i_n$ are different), it is necessary and sufficient that from the function Φ, with all possible substitutions of zeroes and units at the place of all the variables from this subset, one obtain not more than two different functions of the remaining arguments.[2] Later A. V. Kuznetsov noted that this proposition is generalised in case of functions of k-valued logic, if the "zeroes and units" are replaced in them by "the values

1. Reported at the Seminar on Mathematical Logic at the Moscow State University, October 1951. Published in the Works of the Mathematical Institute imeni Steklov 51 (1958), 186 -- 200.

2. A proposition close to this was published in 1954 by G. N. Povarov in reference [1], for the first time although after it was reported by A. V. Kuznetsov.

0, 1,...,k-1," and the words "not more than two" are replaced by the words "not more than k." In the case of a function of infinite-valued logic, all the subsets of the arguments are selectable.

A. V. Kuznetsov[1] proved several propostions concerning the selectable subsets of functions of algebraic logic, including the following:

1) If A and B are partially intersecting (i.e., $A\cap B$, $A\setminus B$ and $B\setminus A$ are not empty) the selectable sets of the arguments of the function Φ, then $A\cap B$, $A\cup B$, $A\setminus B$ and $B\setminus A$ are also selectable.

2) For any function, the selectable sets form a lattice relative to inclusion.

3) If the sets $\{x, y\}$ and $\{y, z\}$ are selectable in a function $\Phi(x, y, z)$, which depends essentially on all its arguments, then $\Phi(x, y, z)$ can be obtained from one of the following functions: xyz, $x\vee y\vee z$, $x+y+z$, by replacing the variables with their negations.

These propositions are not generalized in the case of k-valued logic. Thus, for example, as noted by A. V. Kuznetsov,[2] the function $\Phi(x, y, z, u) = (x+y+z+u^2)^2 \pmod 3$ is such that the sets $\{x, y, u\}$ and $\{y, z, u\}$ are selectable, and their intersection $\{y, u\}$ is not selectable. In the same place A. V. Kuznetsov proves that any two representations of the functions of algebraic logic in the form of non-repeated superpositions through non-separable functions are in a certain sense almost identical and indicates then a way of selecting, from among all such representations, in a definite sense, the canonical ones, which are now found to be unique for each function. Incidentally he considers cases when the function $\Phi(x_1, \ldots, x_n)$ can be represented in the form $\Phi(\varphi(x_{i_1}, \ldots, x_{i_m}), \chi(x_{i_{m+1}}, \ldots, x_{i_n}))$, where all the indices $i_1, \ldots, i_n$ are different, and $\Phi(x, y)$ is one of the following three functions: xy, $x\vee y$, $x+y$. In the case when $\Phi(x, y) = xy$, the function Φ is called a π-function, in the case when $\Phi(x, y) = x\vee y$ it is called a δ-function, while in the case of $\Phi(x, y) = x+y$ it is called a σ-function. A. V. Kuznetsov proves that the set of

1. Published in the Works of the Mathematical Institute imeni Steklov 51 (1958), 205 -- 211.
2. Ibid.

π-functions, the set of δ-functions, and the set of σ-functions do not intersect pairwise. G. N. Povarov [1] proved that any symmetrical function of algebraic logic, different from the functions

$$x_1 x_2 \ldots x_n, \ \bar{x}_1 \bar{x}_2 \ldots \bar{x}_n, \ x_1 + x_2 + \ldots + x_n$$

and of their negations, is non-separable.

7. G. N. Povarov investigated also many other questions concerning symmetrical, monotonic, and related functions of algebraic logic. From the results he obtained we shall mention here the following.[1]

1°. In order for the function $\Phi(x_1, \ldots, x_n)$ to be symmetrical, it is necessary and sufficient that the following two identities be satisfied for it:

$$\Phi(x_1, x_2, \ldots, x_n) = \Phi(x_2, x_1, x_3, \ldots, x_n) = \Phi(x_2, x_3, \ldots, x_n, x_1).$$

The functions Φ and Ψ are called "of the same type" if Φ can be obtained from Ψ and Ψ can be obtained from Φ by substitution of variables of their negations. The single-type relation is of interest, since the single-type function differs little when represented through various functions, among which is $\bar{x}$, and also in the realization by contact circuits or similar circuits, and can be readily replaced by another. G. N. Povarov showed (for functions of algebraic logic):

2°. The number N_n of types of functions of n variables is estimated to be

$$\frac{2^{2^n}}{n!\,2^n} < N_n < (1+\varepsilon)\frac{2^{2^n}}{n!\,2^n},$$

where ε tends to 0 with increasing n.

3°. The number of types of symmetrical functions of n variables is equal (for $n \geqslant 1$)

$$2^n + 2^{\left[\frac{n}{2}\right]} + 2^{\left[\frac{n+1}{2}\right]} - n - 1$$

[2]

4°. The number of functions of n variables, which are of the same type as the symmetrical ones, is equal

1. Published in references [4, 5, 6].
2. Here and henceforth, $[\alpha]$ will denote the integral portion of the number α.

(for $n > 1$) to $2^{2n} - 2^{n+1} + 4$.

8. A. A. Markov in his paper [53] considers the question of estimating the inversion complexity of finite systems of functions of algebraic logic. Here the "inversion complexity" of a system of functions $\Phi_1, \dots, \Phi_m$ is called the least of the numbers s such that there exists a system $\mathfrak{A}_1, \dots, \mathfrak{A}_m$ of representations of given functions in terms of $\bar{x}$, xy and $x \vee y$, in which the number of different negated subformulas (i.e., those directly under a negation sign) is equal to s. A. A. Markov proved that the greatest of the inversion complexities of a system of m functions of n variables is equal to D(n) for $m > 1$ and pd(D(n + 1)) for m = 1, where pd(r) has a value r − 1 when $r > 0$, and a value of 0 when r = 0, while D(r) is the so-called binary dimension of the number r, defined by A. A. Markov as the least of such natural numbers y, that $r < 2^y$ (i.e., $D(0) = 0$, $D(r) = [\lg_2 r] + 1$ for $r > 0$). In addition, A. A. Markov introduced the concept of sign-variability of the function Φ, giving this name to the largest of the possible numbers of reversals of the value of the function Φ upon monotonic change of the values of all its arguments, increased by unity (for a more accurate definition see [53]), and proved the following theorem: the inversion complexity of a function Φ (i.e., of a system consisting of a single function Φ) is equal to $pd(D(\mathrm{Alt}(\Phi)))$, where $\mathrm{Alt}(\Phi)$ is the sign alternation of the function Φ.

9. The problems solved in the Markov paper [53], which is discussed in the previous item, serve as examples of a broad class of so-called problems of minimization, or in general problems of simplification of various means of representation of functions of algebraic logic and estimates of to what extent, in a maximum fashion, they can be simplified in a certain respect or another.[1] By minimization one means precisely the maximum simplification. In means of representation (or in other words, realization) of functions of algebraic logic are meant to be various types of expressions of algebraic logic, relay-contact, or other circuits, etc. The problems about which we speak

1. See A. A. Markov [53], for example, with respect to reducing the number of negations.

have been extant in technology for a long time. The attention of our mathematicians was attracted to them under the influence of many lectures by V. I. Shestakov at the Moscow State University , the lectures of P. S. Novikov on algebraic logic (Moscow State University, fall 1950), who formulated many questions more accurately, and the lectures of G. N. Povarov (at the Moscow State University and the Mathematics Institute imeni Steklov), who investigated the problems connected with the estimates given in the works by Shannon.

From among questions of this kind, the first to attract attention of our mathematicians were questions of simplification and minimization of disjunction normal forms (DNF). Among the Soviet mathematicians, the concept of the abbreviated DNF was first introduced (1951) by a student of P. S. Novikov, S. V. Yablonskiy,[1] who defined in reference [3] this form with the aid of an algorithm by which it is obtained from the perfected DNF, and who applied it to the investigation of monotonic functions. Soon after this, A. V. Kuznetsov engaged in the study of abbreviated DNF, and he, on the basis of another, more direct definition of these forms, illustrated their significance to many problems in the theory of contact circuits, particularly for questions connected with a so-called non-repeating circuit and the problem of generalization of the principle of duality for two dimensional circuits (proving the impossibility of its generalization for non-planar circuits).[2] In connection with this circumstance, ob-

---

1. An analogous concept, differently defined and named, was contained in one paper by Black, lithographed in Chicago in 1938, as turned out later, and it was considered, almost simultaneously with S. V. Yablonskiy, by Quine (published in the U.S.A. in 1952). See Works of the Mathematics Institute imeni Steklov 51 (1958).
2. These results were first reported in the fall of 1952 at the Seminar on Elementary Problems of Mathematical Logic in Moscow State University (see Ibid, pp. 174 -- 185).

served by Quine, that the abbreviated DNF sometimes admits of a further simplification (owing to the fact that some of its terms are absorbed by disjunction of the remaining terms), the concept of minimal DNF came into use. A minimal DNF, and also the related so-called blind-alley DNF engaged the considerable attention of S. V. Yablonskiy and his students (Yu. I. Zhuravlev, L. Yermakova, and others). S. V. Yablonskiy gave a geometric form[1] to the investigations of the DNF (and general functions of algebraic logic), at which each group of n values of the arguments of the function $\Phi(x_1,\ldots,x_n)$ was assigned a vertex of a unit n-dimmensional cube, and to the function itself -- the set E_Φ of all the vertices of the cube, corresponding to those groups of n $(\alpha_1,\ldots,\alpha_n)$, for which $\Phi(\alpha_1,\ldots,\alpha_n)=1$, and to each DNF of the function Φ -- a certain covering of the set E_Φ by a system of so-called intervals of the cube (i.e., vertices, ribs, faces, etc.). The concept of a rank of an interval is introduced, on the problem of minimization reduces to finding such a covering, the sum of the ranks of which is a minimum. With this, he considers also that case, when the function $\Phi(x_1,\ldots,x_n)$ is defined not for all groups of n values of the argument, but only for a certain set A of these and upon suitable simplification of the representation of its expressions by algebraic logic (which equals tó it on A) the latter can be replaced by others, not equal to them outside A (but also equal to the functions Φ on A).

Such functions were considered earlier in several papers by V. N. Roginskiy (see, for example, [1]). S. V. Yablonskiy has adopted for the consideration of such functions the concept of separability of sets, taken over from the descriptive theory of sets (see Sections 2 and 8). Using this constant, Yu. I. Zhuravlev solved completely several problems connected with minimization of such (not everywhere defined) functions [2]. Inciden-

1. These results were first reported in the fall of 1952 at the Seminar on Elementary Problems of Mathematical Logic in Moscow State University (see Ibid. pp. 174 -- 185). p. 23 -- 27 and 143 -- 157 (paper by Yu. I. Zhuravlev.

Incidentally Yu. I. Zhuravlev obtained a criterian which permits recognizing whether a certain system of intervals of the cube covers a certain interval, i.e., whether a given term of a certain given DNF is absorbed by the aggregate of its remaining terms. Quite recently Yu. I. Zhuravlev succeeded in defining a new canonical (in the sense that it is uniquely determined from the function) DNF, which is much simpler in many cases than the abbreviated DNF, but which still is such that all minimal DNF are obtained by its simplification (if it is not minimal).

A new relatively simple algorithm for the construction of an abbreviated DNF, and from it all minimal DNF (and also the analogous conjunctive normal forms (CNF)) was recently constructed by Ye. K. Voyshvillo.[1] In the first part of this paper (the finding of the abbreviated DNF) this algorithm is related to the well-known algorithm of Nelson, but differs substantially in a more rational method of opening the bracket (when going from the CNF to the DNF) with the aid of a unique operation introduced by Ye. K. Voyshvillo, so-called carrying outside the vertical line, used in the second part of the algorithm. The difficulties as well as the possibilities of overcoming these difficulties (in the sense of cumbersomeness and laboriousness), arising in the opening of the brackets with respect to the distributive properties, were investigated earlier also by S. V. Yablonskiy [10] in his joint work with I. A. Chegis [1], devoted to the theory of the so-called tests for electric circuits, i.e., such sets T of groups of n values of variables, which for a given circuit are such, that in order to verify the correctness of the circuit it is enough to verify the function $\Phi(x_1,\ldots,x_n)$ which realizes this circuit on T. To open the brackets he proposed a unique geometric method, based on the use of the so-called sieve, related to the known sieve operation in descriptive theory of sets (see Sections 2 and 8 of the present article). As noted by S. V. Yablonskiy, there is a close relation between the minimization of the tests and

1. Ye. K. Voyshvillo. Method of Simplifying Forms of Expressions of Functions of Truth. Nauchnyye doklady vysshey shkoly, filosofskiye nauki [Scientific Papers of the Higher Schools, Philosophical Sciences] No. 2 (1958).

the minimization of the DNF, and consequently many results pertaining to the construction of tests can be carried over to the construction of minimal and blind-alley DNF.

10. From among the results in the field of minimization connected with estimates of the extent to which it is possible to simplify a given type of representation of a function of some class, one of the first results in our country was the following one of S. V. Yablonskiy. In /_5_/ Yablonskiy gave such a method of representing linear functions of algebraic logic in terms $\bar{x}$, xy and $x \vee y$, for which (in the representation of a function of n variables) the number of (variable) letters is less than $\frac{9}{8} n^2$. Later on V. K. Korobko gave in /_1_/ a method of representing symmetrical functions in terms of $\bar{x}$, xy and $x \vee y$, in which the number of letters is less than

$$9{,}375\,[\lg_2 n] \cdot 2^{([\lg_2 n]^2 + [\lg_2 n])/2}.$$

B. I. Finkov investigated the representations of such functions $\Phi(x_1, \ldots, x_n)$ of algebraic logic, which assume values of 1 only on a small number of collections $(a_1, \ldots, a_n)$ of values of arguments, and showed /_1_/ that for any such function there exists a representation in terms of $\bar{x}$, xy and $x \vee y$, which has not more than $2n + l \cdot 2^{l-1}$ letters.

Investigations of estimates of more general character, concerning the representation (realization) of functions (or other objects, the number of which is estimated) by various means from a sufficiently broad class, were the ground work for the work of O. B. Lupanov /_1, 3_/. For the quite general case considered in them (generalized circuits, constructed from a definite stock of elementary means), O. B. Lupanov gave a lower estimate of the complexity of the minimal representations, which generalizes the well-known estimate of Shannon for the representation of functions of algebraic logic by contact circuits.

R. Ye. Krichevskiy[1] considered a case of representation of the function of k-valued logic of n variables in terms of a function of an arbitrary given finite set, and obtained a generalization to that case of the well-known

---

1. Printed in the collection "Problemy kibernetiki /_Problems of Cybernetics_/, No. 2.

estimate of Riordan and Shannon for representations of the functions of algebraic logic in therms x, xy and $x \vee y$. Indeed, R. Ye. Krichevskiy showed that the number of letter variables in the representation of a certain (poorly representable) function should be greater than $(1-\varepsilon)\frac{k^n}{\lg_k n}$, where ε tends to 0 with increasing n. O. B. Lupanov, obtaining a corresponding upper estimate, recently showed that this estimate is asymptotically exact for k = 2. R. Ye. Krichevskiy considered, on the other hand, a generalization of his estimate to the case of representations obtained by superposition of arbitrary objects of a certain nature. An example of such, differing from the ordinary one, are, for example, superpositions of contact circuits, put into consideration in 1951 by A. V. Kuznetsov[1] and contained also in the paper by B. A. Trakhtenbrot [4].

11. We gave above several examples of this development and generalization, to which algebraic logic is subjected more and more in recent years, particularly under the influence of questions which arise in its applications. This process of development of algebraic logic went far beyond the limits of its previous boundaries (Boolean algebra and propositional functions) and intertwines with numerous other fields of mathematics: general theory of functions, for example, functional constructions in k-valued and infinite-valued logics), general algebra (for example question of functional completeness of universal algebras and the use of the general concept of operations of closure), topology, and combinatorics (theory of contact and other circuits and other means of realization of functions of algebraic logic, superposition of arbitrary objects, etc.). We give a few other examples.

One of such examples is the theory of matrices on Boolean algebra, developed (in connection with the application of circuit theory) principally in the papers by A. G. Lunts [1, 3, 4]. From among the other papers in this field, we mention the work by M. L. Tsetlin [3] and G. N.

1. In the paper delivered to the Seminar on Mathematical Logic in Moscow State University, October 1951.

Povarov [3]. A. V. Kuznetsov proposes[1] to carry out many considerations not in the algebra of matrices on Boolean algebra, but in the algebra that is isomorphic to it of relations, and outlines certain constructions connected with this. G. N. Povarov in his papers [10] and [14] considers matrices on algebras of more general form than Boolean algebra, on so-called numeroids. Here a numeroid is called an algebra by two organizations -- addition and multiplication, of which the multiplication is distributive relative to addition, and both operations are associative and commutative and such that there exists such a unique element 0, that $0+x=x+0=x$, and such a unique element 1, that $1\cdot x=x\cdot 1=x$. Matrices on such numeroids (an example of which is the natural series with ordinary addition and multiplication) and the so-called quasi-minors of these matrices, are used by G. N. Povarov for the analysis of circuits and graphs.

Other examples are the various formalisms that are being built up for problems connected with so-called multi-step relay-contact circuits (or other circuits of relay action, i.e., in general finite automatic machines). In such circuits, the elements (relays) assume states which depend not only on the input variables, but also successively changing with them. In this connection the work of the circuit is described, for example, by a sequence of n-term sets of functions of algebraic logic, i.e., n-dimensional vectors of Boolean algebra. With this, the time is represented by means of an integer parameter, explicitly written our or else simply appearing as the number of the place in a sequence of vectors. It is this particular way that is being followed by several authors. Among such papers we mention those by V. I. Shestakov [8 -- 12, 14].

Shestakov calls sequences of the above kind processes and he investigates both the ways of obtaining the corresponding process from the equations corresponding to the circuit (and to the input process) (system analysis), as well as the reverse -- obtaining the equations from the process (system synthesis).

---

1. At the end of the article in the Works of Mathematics Institute imeni Steklov 51 (1958), 217 -- 220.

Many authors consider the concept of operator, corresponding to a given finite automaton (multiple-contact circuit). This is the operator which converts the input process, which is "given out" by the cirtuit (output process). Questions on a class of all such operators were investigated by Yu. T. Medvedev in [8] and by B. A. Trakhtenbrot in [13]. Continuing this investigation, B. A. Trakhtenbrot came to the conclusion in 1957[1] that it is advisable to use in these questions the formalism of calculus of predicates. With this, the sequence of n-dimensional vectors is replaced by n predicates of one numerical argument. In the operators themselves (realizable in finite automata) are formulas with variable single-place predicates, quantors obtained from them, and limited quantors obtained from object (numerical) variables. B. A. Trakhtenbrot constructed an algorithm, which permits, by means of any such formula (which contains only one free object variable -- the number of the step) to obtain equations which describe the operators specified by this formula.

This is essentially one of the examples of the broadening of algebraic logic towards the calculus of predicates. It is precisely here that the operations of the calculus of single-place predicates with quantors obtained from them, that is being brought into the sphere of algebraic logic. Algebraization of these quantors is facilitated by the fact that the problem of solvability has been solved for this calculus.

This path of expanding algebraic logic is related to another tendency, defined not so much by the need of applications, as by the very development of modern general algebra, particularly the theory of general algebraic systems and in particular the theory of models. This second tendency consists of an algebraic approach to all concepts and operations of the calculus of predicates and the related problems, consist of algebraization of the calculus of predicates itself with subsequent application of the

1. Reported at the Seminar on Mathematical Logic, Moscow State University, 15 May 1957.

resultant formalism to problems in algebra itself. This tendency is found, for example, in many papers of A. I. Mal'tsev, including those which were already treated in Section 12 of this article.

Conclusion

1. As already noted in the introduction, our survey does not pretend to be complete. Many papers pertaining to the field of mathematical logic or its applications were not touched upon here. Not always could we stop to a sufficient degree on the history of the considered problems. We left aside for the time being questions pertaining to mathematical linguistics and the associated problems, arising in the creation of information machines or machines for the translation of one language into another. We did not treat papers devoted to philosophical problems of mathematical logic. We completely avoided the problems related with the history of mathematical logic and mathematics. Certain of these problems we shall attempt, cursorily, to supplement in this conclusion, which is unavoidably therefore quite spotty.

2. Works in the field of mathematical linguistics were carried out principally:

1) In the Division of Scientific Research of Applied Mathematics (OPN) of the Mathematical Institute imeni Steklov under the leadership of A. A. Lyapunov. Participating in this work were O. S. Kulagina, I. A. Mal'chuk, and T. N. Moloshnaya, who developed algorithms for translation from French into English and many theoretical problems connected with their compilation.

2) At the Seminar on Mathematical Linguistics of the Moscow State University, under the leadership of V. V. Ivanov, P. S. Kuznetsov, and V. A. Uspenskiy. Papers were delivered at the seminar also by R. L. Dobrushin, T. N. Moloshnaya, I. A. Mel'chuk, O. S. Kulagina, I. I. Revzin, V. A. Purto, and S. K. Shaumian.

3) In the joint group on machine translation of the First State Pedagogical Institute for Foreign Languages

under the leadership of V. Yu. Rozentsveyg and I. I. Revzin. The joint group began to publish a "bulletin" (seven issues were published), in which are printed stenographic reports of the sessions, and also articles and communications on mathematical linguistics.

4) Problems of machine translation are being developed in the Institute of Precision Mechanics and Computational Technology of the USSR Academy of Sciences and in the Laboratory of Electric Simulations of the All-Union Institute of Scientific and Technical Information.

5) Since 1956, the Leningrad University has had in operation an all-university seminar on the theory of machine translation under the leadership of N. D. Andreyev. Participating in it were staff members of the Mathematical-Mechanical, Philological and Eastern faculties. The seminar engaged in researches both on the general theory of machine translation and on the compilation of particular algorithms.[1]

One of the first to engage in mathematical linguistics in the USSR was A. N. Kolmogorov, who already in the twenties formulated a definition of the case as a class of logical relations that are equivalent to each other (in a definite sense).[2]

G. S. Kulagina (a student of A. A. Lyapunov) formulated [4] the principal premises of set-theoretical concepts of language, which served as the basis for further work in this field.

The concept of O. S. Kulagina is found to be particularly useful in the study of so-called simple languages, which represent, however, only a simplified model of real languages.

Certain definitions, pertaining to the study of

---

1. Note added in proof. In March 1958 there was organized at the Leningrad University an experimental laboratory for machine translation, where these researches are being carried out.

2. See Bulletin of Joint Group on Machine Translation, No. 5, Article by V. A. Uspenskiy [15, 16] and R. L. Dobrushin [16].

non-simple languages, were given by V. A. Uspenskiy and R. L. Dobrushin.

The abstract model, introduced by O. S. Kulagina for the purpose of formalization of certain grammatical categories, was found to be, as frequently happens in mathematics, applicable in a broader sense than was originally thought: as noted by V. V. Ivanov, certain aspects of this model can be used for the construction of the theory of meaning of linguistic expressions; V. A. Uspenskiy made analogous remarks on the use of the constructions of O. S. Kulagina in phonology.

Significant for further research was the concept, introduced by O. S. Kulagina, of the configuration (which in some sense is the refinement of the concept of "word combination"). Roughly speaking, a configuration is such a finite sequence of elements, which, without disturbing the understandability, can be replaced by a single element (resultant). With such a replacement ("abbreviation") the length of the phrase is naturally reduced. In order to establish a definite sequence of abbreviations, which must be carried out in a phrase, the configurations are classified by ranks.

I. I. Revzin [2, 3] applied the theory of configurations to the formalization and refinement of several concepts of traditional syntax; it was found here that in the construction of a syntax in terms of configurations it is impossible to bring into consideration such phrases as "it freezes" (so-called "impersonal" clauses) in general any single-element clauses.

3. In connection with the problem of constructing information machines (for such sciences, for example, as chemistry, where the volume of information is particularly large), capable not only of storing and issuing information, but also by logically processing the information, the creation of an artificial "machine language" becomes quite urgent. We speak here of the construction of a formalized language with an exact indication of its contentful interpretation and constructive definition of intelligence (and not only regular construction) of the phrases of this language. The principles of such a language -- algorithms

of definition of intelligence of its phrases for the simplification of a particular branch of chemistry (synthetic organic chemistry) have been constructed by V. K. Finn[1] together with D. G. Lakhuti and G. E. Vleduts. D. G. Lakhuti and V. K. Finn, on the basis of these works, attempt to formulate the principle of constructive logical semantics.

4. Researches connected with mathematical logic in the field of general logic were the subject of papers delivered at the Seminar on Logic at the Institute of Philosophy Academy of Sciences, USSR. Inasmuch as the papers of A. A. Zinov'yev[2] contain detailed information on the work of this seminar, we permit ourselves to note here only that the seminar paid great attention to problems of logical analysis of knowledge of connections (works of A. A. Zinov'yev, V. K. Finn, and D. G. Lakhuti) and is different from the simpler knowledge of relations.

Concerning various views on modern mathematical logic, a paper was delivered by S. A. Yanovskaya (20 December 1956) on philosophical readings, at the Institute of Philosophy Academy of Sciences USSR, the contents of which is reported in the note by D. G. Lakhuti [1] and N. I. Styazhin [3].

Problems of philosophical character, pertaining to mathematical logic, were dealt with also by the students of S. A. Yanovskaya: B. V. Biryukov, A. D. Getmanov. B. Yu. Pil'chak [2] and N. I. Styazhkin [1, 2, 3]. The remarks of N. I. Styazhkin [2, 3] are devoted to an interpretation of logical antinomies, as evidencing the invalidity of certain idealizing assumptions (for example, the fact that objects of a considered object region represent absolutely solid unchangeable bodies, retaining their individuality as they are included in any set of objects). Emphasizing that "dialectical materialism requires resolution and not fetishization of contradictions" and that "dialectical contradiction has nothing in common with formal-logical contradiction, i.e., contradiction of such a kind, for example,

1. See the paper by A. A. Zinov'yev in "Voprosy filosofii" [Problems of Philosophy], 2, (1958).
2. See A. A. Zinov'yev [1]; see also his paper in "Voprosy filosofii", 2 (1958).

as a quantity which is both zero and not zero simultaneously and in other words, quietly bearing with such a delicate situation" ([3], p. 92) Styazhkin notes the following. Styazhkin notes that the resolution of the antinomy is attained not by a simple foregoing of coarsening propositions, but the refinement of such, based on dialectical-materialistic principle of concreteness of truth. The new contradictions that arrive thereby are resolved in turn with the aid of further refinement, and thus, the problem of formalization of any contentful theory leads to the need for considering this theory in its variation and development, ie.e, from the point of view of the dialectical logic.

Devoted to a criticism of the attempt of logicism (Russell and his followers) to reduce mathematics to logic is an article by A. D. Getmanova. The success of the logical-mathematical calculus constructed by Russell and Whitehead in Principia Mathematica is particularly instructive from the point of view of dialectical materialism, because it was indeed with the aid of Principia Mathematica that further development of science has proved the unrealizability of the premises of Russell that mathematics can be reduced to logic. Actually, while the purely logical part of systems of the type of Principia Mathematica (the K_0 calculus of D. A. Bochvar) is reducible to the narrow calculus of predicates (with equality), for which the Goedel theorem on completeness is true, the entire system as a whole (including the arithmetic of natural numbers with recursive functions) is known to be incomplete (and incompletable) (the Goedel theorem on incompleteness). In other words, while logical constants (negation "no," conjunctions "and," "or," "if...then," quantors "all" and "exists," and the identity relation) are uniquely defined (in models) by axioms and rules of the purely logical part of calculi of the Principia Mathematica type, it follows from the Goedel theorem on incompleteness that it is impossible to define uniquely arithmetic terms by means of logic even through the mathematization of the latter, realized in fact by Russell and Whitehead.

B. V. Biryukov dealt with the clarification of the logical-mathematical work of G. Frege (particularly his

theory of sense and meaning, and also his fight against subjectivism in logic).

5. Certain problems of the history of the creation and development of mathematical logic have been the topic of communications and papers by G. Ruzavin and N. I. Styazhkin. N. I. Styazhkin [1] engaged particularly with the history of mathematical logic in pre-revolutionary Russia (P. S. Poretskiy, S. O. Shatunovskiy, I. V. Sleshinskiy, Ye. L. Bunitskiy, N. A. Vasil'yev, M. S. Volkov, N. N. Parfent'yev, and others), the semantic paradoxes among the middle age scholasticists, and an analysis of Boole's logical ideas. It was found in particular that although the algebraic logic constructed by Boole actually was closer to a Boolean ring than to a Boolean algebra, it nevertheless is neither one directly, but the secret of the success of the methods used by Boole himself requires explanation (which is indeed noted by N. I. Styazhkin.)

With respect to the semantic paradox of the type "liar" (for example something like the following: in a volume are written only two propositions p_1 and p_2. p_1 is a fixed true statement, and p_2 says "only p_1 is true"; it is required to determine whether p_2 is true or is it false), considered by Albert of Saxony and John Buridan (16th century), an interesting observations is made by N. I. Styazhkin on how Buridan eliminates paradoxes by using the so-called "paradoxical consequences," (which are obtained, for example, by adding auxiliary premises of the type "Socrates said" and the interpretation of a paradox as denoting merely that Socrates could not have said such a phrase).

6. General problems of axiomatics and its history were treated by Yu. A. Rozhanskaya and S. A. Yanovskaya.

In note [15] (1953), Yu. A. Rozhanskaya, leaning on concepts she introduces, those of x-equivalence (certain generalized quantor concepts of similarity of two sets) and x-type, proves (purely set-theoretically) the equivalence of the following two definitions of completeness of the system:

(A) A system of axioms is complete if one cannot

add to it a single axiom, expressed in terms of the same relations, without stopping it either from being compatible or from being independent.

(B) A system of axioms is complete, if any two of its interpretations are isomorphic.

Devoted to the question of the history of axiomatics was a paper by S. A. Yanovskaya at the Third All-Union Mathematical Congress [29]. The paper considers the question of why geometry, even in Euclid's time, was constructed axiomatically, whereas an axiomatic construction of the arithmetic came into practical use only from the time of Peano (i.e., the end of the 19th century). The hypothesis proposed by Yanovskaya as an answer to this question consists, roughly speaking[1], of the following. The discovery of the irrationality was not the consequence of axiomatic construction of geometry, but to the contrary, it preceded the latter. The basis of this discovery lies in the Pythagorean theorem, empirically observed by operating with a compass and rule, and in the assumption, based on the idealization of the same operation, of the existence of an ideally exact square. However, this discovery invited the conclusion that the geometric problems can best be solved not by calculation, but by construction. The point is that, just like arithmetic, geometry was needed above all as a set of instructions for action, as an operator science, which has constructively developed general methods (algorithms) of solving entire classes of homogeneous geometrical problems (or the "mass problem" corresponding to each such class: such as, for example, "divide (an arbitrary!) segment in two"). But unlike the algorithms of arithmetic, where one always proposes a potential realizability of any natural number, algorithms that solve the

1. For a detailed report see Istoriko-matematicheskiye issledovaniya [Historical-mathematical Researches], XI. The motivation of this hypothesis required the use of so much historical-mathematical and historical-philosophical material, that the author had to forego the clarification of other questions indicated in [29].

mass construction problems depend specifically on the instruments that can actually be used.

The problem of solving a problem of construction cannot even be formulated, if one does not agree beforehand what instruments can be used: what operations are proposed to be directly realizable, although in practice, possibly, they are not always such. Solution of the problem consists thus of reducing it to problems that are assumed to be solved, and the algorithm, which solves the mass geometric problem by construction is already, unlike the algorithms of the arithmetic of natural numbers, a reducibility algorithm. The postulates of Euclid indeed formulate exactly the problems that are assumed by him to be already solved.[1] In particular, the fifth postulate discusses when one can consider the problem of finding the point of intersection between two lines solved. Euclid did not accidentally formulate this postulate in such a way, that the criterion that recognizes whether the problem of finding the point of intersection of two given (arbitrary!) straight lines is solvable has in a certain sense the following effective character: recognize whether the condition of this criterion (pertaining to a sum of certain two angles) is satisfied, possibly by means of simple constructions with a compass and rule in a limited portion of a plane.

But after the postulates were already formulated, and thus in such a way that one should have begun the exposition of geometry from them, the conversion of the latter into a deductive axiomatic theory could become and did become a natural successive stage in the history of geometry.

The development of geometry as a deductive science, unlike arithmetic which remains essentially an operative

1. It goes without saying that the problems considered solved were not arbitrary, but only those which persons frequently encountered in practice and for the solution of which (in ordinary cases) correspondingly instruments were already available.

science,[1] was thus due to the peculiar operative character of geometry: to the circumstance that the algorithms of the "principles" of Euclid are not absolute, but reducibility algorithms.

The difficult problems connected with the axiomatic theories, such as the question of methods of recognition of derivability or non-derivability of anything in such a theory, requires for its solution the development of a general theory of algorithms. But even problems in the history of axiomatics are made clearer to some extent in the light of the theory of algorithms.

1. In the "Principles" arithmetic is included in geometry, since arithmetic operations are realizable with a compass and rule. However, as shown by I. G. Bashmakova [1], the arithmetic of abstract numbers is proposed by Euclid in his construction of the theory of measure-numbers, which are constructed with a compass and rule.

Bibliography

Adyan, Sergey Ivanovich

1. Algorithmic Unsolvability of Problems of Recognition of Certain Properties of Groups, "Doklady Aka. Nauk SSR [Trans. Acad. Sci. USSR] 103, (1955), 533-535.

2. Problem of Divisibility in Semigroups, "Doklady Akad. Nauk SSSR, 103, (1955), 747-750.

5. Finite-Definite Groups and Algorithms, "Uspekhi Matematicheskikh Nauk, UMN [Progress in Mathematical Sciences] 12:3 (75), (1957), 248-249.

6. Role of Reduction Law in Specifying Semigroups with Reduction by means of Defining Relations. "Doklady Akad. Naus SSSR, 113, (1957), 1191-1194.

Arin', Eyzhen Indrikovich

3. On One Generalization of the Baire Theorem. "Uspekhi Mat. Nauk," 8:3 (55) (1953), 105-108

Arsenin, Vasiliy Yakovlevich

4. Theory of A sets, "Uspekhi Nauk" 5:5 (39), (1950), 45-108 (with A. A. Lyapunov)

Bashmakova, Izabella Grigor'yevna

1. Arithmetic Books, The Euclid "Principles." Ist.-matem. issledovaniya [Historical-Mathematical Investigations], 1 (1948), 296-328.

Bochvar, Dmitriy Anatolevich

4. Concerning the Problem of Paradoxes of Mathematical Logic and Set Theory. Matematicheskiy sbornik [Mathematics Collection] 15(57) (1944), 369-384

6. On the Problem of Paradoxes and on the Problem of Broadened Calculus of Predicates. ibid. 43(84), (1957) 3-10.

Chegis I. A.

1. On Tests for Electric Circuits, Uspekhi Matematiches Nauk 10:4 (66((1956), 182-184. (Togethar with S. V. Yablonskiy).

Detlovs, Vilnis Karlovich

1. Normal Algorithms and Recursive Functions. Dokl. Akad. Nauk SSSR 90(1953), 723-725.

Finikov Boris Ivanovich

1. On One Family of Classes of Functions of Algebraic Logic and Their Realization in the Class of π-Networks. Doklady Akad. Nauk SSSR 115(1957), 247-248

Glivenko, Valeriy Ivanovich

4. Sur la logique de M. Brouwer, Bull. Acad. Sci de Belgique (5), 14 (1928), 480-483.

5. Sur quelques points de la logique de M. Brouwer, Bull. Acad. Sci. de Belgique (5), 15 (1929), 183-188.

Kolmogorov, Andrey Nikolayevich

8. On the Excluded Third Erinciple, Matematicheskiy Sbornik, 32 (1925), 646-667

34. Zur Deutung der Intuitionistischen Logik, Math. Z. 35(1932) 58-65

135. On the Concept of the Algorithm. Uspekhi Matematicheskikh Nauk 8:4 (56) (1953), 175-176

Korobkov, V. K.

1. Realization of Symmetric Functions in a Class of π-Networks. Doklady Akad. Nauk SSSR, 109 (1956), 260-263.

Kozlova, Zoya Ivanova

1. On the Question of Approximate Solution of Partial Differential Equations with the Aid of Electric Models. Avtomatika i Telemekhanika [Automation and Remote Control] 17 (1956), 890-896 (with N. S. Nikolayev)

1. On Multiple Separability, Doklady Akad. Nauk SSSR, 27 (1940), 108-111

2. On Certain Classes of A and B Sets. Izvestiya Akad. Nauk SSSR Ser. Mat. 4 (1940), 479-500.

3. Principal Theorems of Multiple Separability. Stalingrad, Uchen. zap. Ped. in-ta [Stalingrad, Scientific Notes of Pedagogical Institute] 1 (1948), 156-168.

5. On Coverings of Certain A Sets. Izvestiya Akad. Nauk SSSR, ser. Mat. 14 (1950), 421-442.

6. Coverings of A Sets and Splittings of B Sets. Uspekhi Matematicheskikh Nauk 5:3 (37) (1950) 130-131

8. Interrelationships Between Theorems of Multiple Separability. Izvestiya Akad. Nauk SSSR, ser. Mat. 16 (1952), 389-404

9. Interdependence of Theorems of Multiple Separability. Stalingrad, Uchen. zap. Ped. in-ta 3(1953), 17-41

11. On the Covering of Sets, Izvestiya Akad. Nauk SSSR, ser. mat. 19 (1955), 125-132.

14. On the Covering of Sets, II, II. Izvestiya Akad. Nauk SSSR, ser. mat. 21 (1957), 349-370

Kreynin, Yakov Lazarevich

1. On Sets That Are Effectively Different from All Φ-Sets. Matematicheskiy sobrnik 38(80) (1956), 129-148

Kulagina, Ol'ga Sergeyevna

4. Concerning One Method of Determining Linguistic Concepts. Byull. ob"yedineniya po problemam mash. perevoda [Bulletin of Joint Group on Problems of Machine Translation] 3 (1957) 1-17

Kutnetsov, Aleksandr Vladimirovich

1. On Primitive Recursive Functions of Large Spread. Doklady Akad. Nauk SSSR 71 (1950), 233-236

2. Investigation of Partially-Recursive Operators by Means of

Theory of Baire Space, Doklady Akad. Nauk SSSR, 105 (1955), 897-900 (With B. A. Trakhtenbrot).
3. On Problems of Identity and Functional Completeness for Algebraic Systems. Trudy 3-go Vsesoyuznogo Matem. S"yezda Vil. II, Moscow (1956), 145-146
4. Certain Problems of Mathematical Theory of Contact Circuits. Topics of Papers Held at the All-Union Conference on Theory of Relay Devices in Moscow. Moscow (1957), 40-41.
5. Completeness of a System of Axioms of Arithmetic with a Rule for Constructive-Infinite Induction. Uspekhi Matematicheskikh Nauk 12:4 (76) (1957), 218-219.

Lakhuti, Delir Gasemovich
1. On Different Views Concerning Modern Mathematical Logic. Voprosy Filozofii, 3 (1957), 208-211 (together with N. I. Styazhkin)

Lunts, Aleksandr Grigorovich
1. Application of Matrix Boolean Algebra to the Analysis and Synthesis of Relay-Contact Networks. Doklady Akad. Nauk SSSR 70 (1950), 421-423
3. Reduction of a Matrix to the Jordan Normal Form. Leningrad, Uchen. zap un-ta, 144, ser. matem. [Leningrad, Scientific Notes of the University, 144, Mathematics Series] 23 (1952), 35-46
4. Algebraic Methods of Analysis and Synthesis of Contact Networks. Izvestiya Akad. Nauk SSSR, ser. mat. 16 (1952) 405-426

Lupanov, Oleg Borisovich
1. Possibility of Synthesis of Networks of Different Elements. Doklady Akad. Nauk SSSR, 103 (1955), 561-563
3. On the Possibility of Network Synthesis with Arbitrary Elements. Trudy 3-go Vsesoyuznogo Matem. S"yezda, Vol. II, Moscow (1956), 146.

Luzin, Nikolay Nikolayevich
90. Lektsii ob analiticheskikh mnozhestvakh i ikh prilozheniyakh [Lectures on Analytic Sets and Their Application] Moscow (1953), 1-360

Lyapunov, Aleksey Andreyevich
17. On R Sets, Doklady Akad. Nauk SSSR, 58 (1947), 1887-1890
19. Theory of R Sets. Uspekhi Matematicheskikh Nauk, 2:3 (19) (1947), 191 plus
24. New Definition of Certain Classes of Sets. Doklady Akad. Nauk SSSR 59 (1948), 847-848
26. On the Effective Measurability Izvestiya Akad. Nauk SSSR, 13 (1949), 357-362
27. On δs-Operations that Retain Measurability and the Baire Property. Matematicheskiy Sbornik 24 (66) (1949), 119-127

28. On Set-Theoretical Operations that Retain Measurability, Doklady Akad. Nauk SSSR 65 (1949), 609-612
29. On Operations That Lead to Measurable Sets, Uspekhi Matematicheskikh Nauk 4:3 (31) (1949), 125 plus
32. Theory of Λ Sets, Uspekhi Matematicheskikh Nauk 5:5 (39), (1950), 45-108 (together with V. Ya. Arsenin)
33. B-Functions, Uspekhi Matematicheskikh Nauk 5:5 (39) (1950), 109-119
36. On Degeneracy Properties for R Sets Izvestiya Akad. Nauk SSSR, ser. Mat. 17 (1953), 563-578
37. On Classifications of R Sets, Matematicheskiy Sbornik 32 (74) (1953), 255-262
38. Separability and Inseparability of R Sets, Matematicheskiy sbornik, 32 (74) (1953), 515-532
39. R Sets, Trudy Matem. in-ta AN [Works of the Mathematics Institute of the Academy of Sciences] 40 (1953), 1-68
44. On Operations on Sets, Admitting of Transfinite Indices, Trudy 3-go Vsesoyuznogo Matem. S"yezda, Vol. II, Moscow (1956), 124-125
45. On the Expansion of Set-Theoretical Operations, Uspekhi Matematicheskikh Nauk, 11:5 (71) (1956), 237
46. On Operations on Sets, Admitting of Transfinite Indices, Uspekhi Matematicheskikh Nauk, 11:1 (67) (1956), 243-244.
51. On Operations on Sets, Admitting of Transfinite Indices, Moscow, Trudy Matem. o-va [Works of the Mathem. Society], 6 (1957), 195-230

Mal'tsev, Anatoli Ivanovich

1. Untersuchungen aus dem Gebiete der mathematischen Logik Matematicheskiy Sbornik 1 (43) (1936), 323-326
7. Concerning One General Method of Obtaining Local Theorems in Group Theory. Ivanovo, Uchen. zap. Ped. in-ta, fiz.-mat. fak-t [Ivanovo, Scientific Notes of the Pedagogical Institute Physical-Mathematical Faculty] 1:1 (1941), 3-9
32. Concerning Pre-Ordered Groups. Trudy Matem. in-ta AN 38 (1951), 173-175
42. Two Remarks on Nilpotent Groups, Matematicheskiy Sbornik 37 (79) (1955), 567-572
45. Quasi-Primitive Classes of Abstract Algebras, Doklady Akad. Nauk SSSR 108 (1956), 187-189
44. On the Representation of Models, Doklady Akad. Nauk SSSR 108 (1956), 27-29
46. Sub-Straight Products of Models Doklady Akad. Nauk SSSR 109 (1956), 264-266
51. On Derivative Operations and Predicates. Doklady Akad. Nauk SSSR 116 (1957), 24-27
52. On Classes of Models with Operation of Generation Doklady Akad. Nauk SSSR 116 (1957), 738-741

Markov, Andrey Andreyevich

30. Impossibility of Certain Algorithms in the Theory of Associative Systems. Doklady Akad. Nauk SSSR 55 (1947), 587-590

36. On Representation of Recursive Functions Doklady Akad. Nauk SSSR 58 (1947), 1891-1892

37. On the Dependence of the Axiom B6 on Other Axioms of the Bernays-Goedel System. Izvestiya Akad. Nauk SSSR, ser. Mat. 12 (1948), 569-570

39. On Representation of Recursive Functions, Izvestiya Akad. Nauk SSSR, ser. mat. 13 (1949), 417-424

40. Constructive Logic, Uspekhi Matematicheskikh Nauk 5:3 (37) (1950), 187-188

41. Theory of Algorithms. Trudy matem. in-ta AN, 38, 1951, 176-189

43. Impossibility of Algorithms of Recognition of Certain Properties of Associative Systems. Doklady Akad. Nauk SSSR 77 (1951), 953-956

44. On Certain Unsolvable Problems Concerning Matrices. Doklady Akad. Nauk SSSR 78 (1951), 1089-1092

46. On Unsolvable Algorithmic Problems, Matematicheskiy Sbornik, 31 (73) (1952), 34-42

48. Theory of Algorithms. Trudy Matem, in-ta AN, 42 (1954), 1-376

49. On the Continuity of Constructive Functions. Uspekhi Matematicheskikh Nauk 9:3 (61) (1954), 226-230

51. On One Principal of Constructive Mathematical Logic Trudy 3-go Vsesoyuznogo matem. s"yezda, Vol. II, Moscow (1956), 146-147

53. Inversion Complexity of Systems of Functions Doklady Akad. Nauk SSSR, 116 (1957), 917-919

Matskina, Roza Yudovna

1. On Continuous Mappings of Hilbert Space, Uspekhi Matematicheskikh Nauk 4:5 (33) (1949), 179

2. On the Complete Universality of the Space of Irrational Numbers. M., Uchen. zap. Obl. ped. in-ta, 15; Trudy kafedry fi.-mat. fak-ta [Moscow, Scientific Notes of the Oblast Pedagogical Institute, 15; Works of the Chair of the Physical-Mathematical Faculty] 1 (1950), 137-140

3. On Continuous Images of Hilbert Space. Izvestiya Akad. Nauk SSSR, ser. Mat. 15 (1951), 95-102

4. Universal Continuous Mapping of Hilbert Space. Izvestiya Akad. Nauk SSSR ser. mat. 15 (1951), 533-544

5. On Mutually-Unique Continuous Images of Hilbert Space. Izvesitya Akad. Nauk SSSR ser. Mat. (1955) 267-272

6. Universal Continuous Mapping of Baire Space on Itself. Glazov, Uchen. zap. Ped. in-ta [Glazov, Scientific Notes of

the Pedagogical Institute], 3 (1956), 100-107

7. Continuous Mappings of Hilbert Space. Trudy 3-go Vsesoyuznogo matem. s"yezda, Vol. II, Moscow (1956), 135

Medvedev Yuriy Tikhonovich

4. On Non-Isomorphic Recursive Denumerable Sets, Doklady Akad. Nauk SSSR 102 (1955), 211-214

5. Degrees of Diffuculty of Mass Problems. Doklady Akad. Nauk SSSR 104 (1955), 501-504

6. On the Concept of the Mass Problem and Its Applications in the Theory of Recurwive Functions and Mathematical Logic Trudy 3-go Vsesoyuznogo Matem. S"yezda, Vol. I, Moscow (1956), 183

8. On Classes of Events, that Admit of Representation in a Finite Automation. In Collection "Avtomaty" [Automata] (Appendix 2) Moscow (1956).

Muchnik, Al"bert Abramovich

1. Unsolvability of the Problem of Reducibility of the Theory of Algorithms. Doklady Akad. Nauk SSSR 108 (1956) 194-197

2. On the Separability of Recursive-Denumerable Sets. Doklady Akad. Nauk SSSR 109 (1956), 29-32

3. Solution of the Post Convergence Problem. Trudy 3-go Vsesoyuznogo matem. s"yezda, Vol. I, Moscow (1956), 184

Nagornyy, Nikolay Makarovich

1. On the Reinforcement of the Reduction of the Theory of Algorithms. Doklady Akad. Nauk SSSR 90 (1953) 341-342

2. On Certain Generalizations of the Concept of Normal Algorithm. Trudy 3-go Vsesoyuznogo matem. s"yezda, Vol. II (Moscow) (1956), 147

3. On the Minimum Alphabet of Algorithms on a Given Alphabet. Trudy matem. in-ta AN, 52 (1957)

4. Certain Generalizations of the Concept of Normal Algorithm. Trudy matem. in-ta AN 52 (1957)

Nivikov, Petr Sergeyevich

8. Sur la separabilite des ensembles projectifs du seconde classe. Fund. Math 25 (1935) 459-466

10. On the Mutual Relationship of Second Class Projective Sets and Projections of Uniform Analytic Complements. Izvestiya Akad. Nauk SSSR, ser. mat. (1937) 231-252

17. On the Consistency of Certain Logical Calculus. Matematicheskiy sbornik 12 (54) (1943) 231-261

18. On Logical Paradoxes. Doklady Akad. Nauk SSSR 56 (1947), 451-453

25. On the Non-Contradiction of Certain Premises of Descriptive Theory of Sets. Trudy matem. in-ta AN, 38 (1951), 279-360

27. On the Algorithmic Inseparability of the Identity Problem, Doklady Akad. Nauk SSSR 85 (1952), 709-712

29. The Unsolvability of the Problem of Conjugation in the

Theory of Group. Izvestiya Akad. Nauk SSSR ser. mat. 18 (1954), 485-524

30. On the Algorithmic Unsolvability of the Problem of Identity of Words in Group Theory. Trudy matem. in-ta AN, 44, (1955) 1-444

Ochan', Yuriy Semenovich

13. Theory of Operations on Sets. Uspekhi Matematicheskikh Nauk 1o:3 (65) (1955), 71-128

Pil'chak, B. Yu.

2. On the Role of the Law of Excluded Third in Mathematics. Novozybkov, Uchep. zap. Ped. in-ta, [Novozybkov, Scientific Notes of the Pedagogical Institute] 1 (1952), 115-131

3. On the Calculus of Problems UMZh [Ukrainian Mathematical Journal] 2 (1952), 174-194

Povarov, Gelliy Nikolayevich

1. On the Functional Separability of Boolean Functions. Doklady Akad. Nauk SSSR 94 (1954), 801-803

3. Matrix Methods of the Analysis of Relay Contact Circuits from the Conditions of Non-Operation. Avtomatika i telemekhanika 15 (1954), 332-335

4. Mathematical Theory of the Synthesis of (1,k)-Pole Networks. Doklady Akad. Nauk SSSR 100 (1955), 909-912; 102 (1955), 196

5. On the Study of Symmetrical Boolean Functions from the Point of View of the Theory of Relay Contact Networks. Doklady Akad. Nauk SSSR 104 (1955), 183-185

6. On a Procedure for the Analysis of Symmetrical Contact Networks. Avtomatika i telemekhanika 16 (1955), 364-366

10. On Certain Matrix Methods of the Analysis of Relay-Contact Networks, Sb. po avtomatike i telemekhanike [Collection on Automatics and Telemechanics] Moscow (1956), 278-285

11. Concerning the Structural Design of Symmetrical Contact Networks. Avtomatika [Automation] 4 (1956), 48-53

14. On Matrix Analysis of Connections in Partially-Oriented Graphs. Uspekhi Matematicheskikh Nauk 11:5 (71) (1956), 195-202

Revzin, I. I.

1. Certain Linguistic Problems of Machine Translation. Voprosy Yazykoznaniye [Problems of Linguistics] 1 (1957) (together with Moloshnaya, Lurks, and Rozentsveyg)

2. Certain Problems of Formalization of Syntasis, Byull. Ob"yedineniya po problemam mash. perevoda 1 (1957), 5-36; 3 (1957), 20-29

Roginskiy, Vadim Nikolayevich

1. Calculation of the Unused States in the Synthesis of Relay-Contact Networks. Avtomatika i telemekhanika 15 (1954), 206-222

Rozhanskaya, Yuliya Antonovna

15. On the Equivalence of Two Definitions of Completemess of the System of Axioms of A. N. Kolmogorov and H. Weyl M. Uchen. zap. un-ta, Vyp. 181, Matematika [Moscow, Scientific Notes of the University, No. 181; Mathematics], 8 (1956), 197-198

Shanin, Nikolay Aleksandrovich

11. On Certain Operations on Logical-Arithmetic Formulas Doklady Akad. Nauk SSSR 93 (1953), 779-782

12. On Submersion of the Classical Logical-Arithmetical Calculus in Constructive Logical-Arithmetic Calculus. Doklady Akad. Nauk SSSR 94 (1954), 193-196

13. On Certain Logical Problems of Arithmetic. Trudy matem. in-ta AN, 43, (1955), 1-112

14. Certain Problems of Mathematical Analysis in Light of Constructive Logic. Z. fur math. Logik and Gruudloger der Mathemakh, 2 (1956), 27-36

15. On Constructive Concept of Mathematical Judgment. Trudy 3-go Vsesoyuznogo matem. s"yezda, Vol. I, Moscow (1956) 189-190

16. On Constructive Mathematical Analysis. ibid. Vol. II, Moscow (1956), 69-71

Shestakov Viktor Ivanovich

8. On the Transformation of Monocyclic Sequences into Reverse Ones. Doklady Akad. Nauk SSSR 98 (1954), 541-544

9. Algebraic Method of Synthesis of Multiple-Step Relay Systems. Doklady Akad. Nauk SSSR 99 (1954), 987-990

10. Algebraic Method of the Analysis of Autonomous Systems of Two-Position Relays. Avtomatika and telemekhanika 15 (1954), 107-128

11. Algebraic Method of Synthesis of Autonomous Systems of Two-Position Relays. ibid. 15 (1954), 310-324

12. Vector-Algebraic Method of Analysis and Synthesis of Multi-Step Relay Systems. Trudy 3-go Vsesoyuznog matem. s"yezda, Vol. I, Moscown(1956), 190-191

13. Algebraic Method of Synthesis of Multi-Beat Systems of r-Position Relays. Doklady Akad. Nauk SSSR 112 (1957), 62-65

Shchegol'kov, Yevgeniy Alekseyevich

2. Elements of the Theory of B Sets. Uspekhi Matematicheskikh Nauk 5:5 (39) (1950), 14-44

Shura-Bura, Mikhail Romanovich

2. Elements of the Theory of B Sets. Uspekhi Matematicheskikh Nauk 5:5 (39) (1950) 14-44

Sodnomov, Bazar Sodnomovich

4. Non-Contradiction of Projective Estimates of Certain Ineffective Sets. Uspekhi Matematicheskikh Nauk 1o:1 (63) (1955), 155-158

7. Non-Contradiction of the Projectiveness of Certain

Remarkable Sets. Trudy 3-go Vsesoyuznogo matem. s"yezda, Vol. I, Moscow, (1956), 184-185

Stupina, Ideya Dmitriyevna

1. On Certain Properties of Γ-Operations. Izvesitya Akad. Nauk SSSR ser. matem. 21 (1957), 329-348

2. On Certain Properties of Λ2 Operations. ibid. 21 (1957), 579-594

Styazkhin, Nikolay Ivanovich

1. On the Problem of the Contribution of N. S. Poretskiy to the Development of Mathematical Logic. M. Vestnik un-ta, ser. ekon., fil. i prava [Moscow, Herald of the University, Series for Economics, Philosophy and Law] 1, (1956), 103-109

2. On the Dialectric Nature of Essence and the Method of Elimination of Paradoxes of Logis. ibid. 4 (1957), 87-90

3. On Different Views on Modern Mathematical Logic. Voprosy filozofii 3 (1957), 208-211 (together with D. G. Lakhuti)

Tartakovskiy, Vladimir Abramovich

26. Explicit Formulas for Local Expansions of Solutions of a System of Ordinary Differential Equations. Doklady Akad. Nauk SSSR 72 (1950), 633-636

Taymanov, Asan Dabsovich

1. On Quasi-Components of Non-connective Sets. Matematicheskiy sbornik 25 (67) (1949) 367-386

2. On Rigid Basis of δ-Operations. Izvesity Akad. Nauk SSSR ser. mat. 14 (1950), 443-448

3. On Quasi-Components of Non-connective Sets. II. Matematicheskiy sobrnik 30 (72) (1952), 465-482

8. On Closed Mappings. I. Matematicheskiy sbornik 36 (78) (1955), 349-352

Trakhtenbrot, Boris Avraamovich

1. Impossibility of an Algorithm for the Problem of Solvability on Finite Classes. Doklady Akad. Nauk SSSR 70 (1950), 569-572

2. On Recursive Separability, Doklady Akad. Nauk SSSR 88 (1953) 953-956

3. Tabular Representation of Recursive Operators. Doklady Akad. Nauk SSSR 101 (1955), 417-420

4. Synthesis of Non-repeated Networks. Doklady Akad. Nauk SSSR 103 (1955), 973-976

5. Investigation of Partially Recursive Operators by Means of the Theory of Baire Space. Doklady Akad. Nauk SSSR 105 (1955), 897-900, (together with A. V. Kuznetsov)

6. Simulation of functions on Finite Classes. Penza, Uchen. Zap. Ped. in-ta [Penza, Scientific Notes of the Pedagogical Institute] 2 (1955), 61-78

11. Determination of a Finite Set and Deductive Incompleteness of the Theory of Sets. Izvestiya Akad. Nauk SSSR ser. mat. 20 (1956), 569-582

12. Signalling Functions and Tabular Operators. Penza, Uchen zap. Ped. in-ta [Penza, Scientific Notes, Pedagogical Institute] 4 (1956), 75-87
13. Concerning Operators that are Realized in Logical Networks. Doklady Akad. Nauk SSSR 112 (1957), 1005-1007

Tseytin, Grigoriy Samuilovich

1. On the Cauchy Theorem in Constructive Analysis. Uspekhi Matematicheskikh Nauk 10:4 (66) (1955), 207-209
2. Concerning the Problem of Recognition of Properties of Associative Calculi, Doklady Akad. Nauk SSSR 107 (1956) 209-212'
3. Associative Calculus with Unsolvable Problem of Equivalence. Doklady Akad. Nauk SSSR 107 (1956), 370-371
5. Theorem of Included Segments, the Cauchy Theorem, and the Rolle Theorem in Constructive Analysis. Trudy 3-go Vsesoyuznogo matem. s"yezda, Vol. I, Moscow (1956), 186-187
7. Uniform Recursiveness of Algorithmic Operators on General-Recursive Functions and Canonical Representation for Constructive Functions of Real Argument. Trudy 3-go Vsesoyuznogo matem. s"yezda, Vol. I, Moscow (1956), 188-189

Tsetlin, Michail L"vovich

3. Application of Matrix Calculus to the Synthesis of Rayleigh Contact Circuits. Doklady Akad. Nauk SSSR 86 (1952), 525-528

Uspenskiy Vladimir Andreyevich

3. On the Concept of Algorithmic Reducibility. Uspekhi Matematicheskikh Nauk 8:4 (1956) (1953), 176
5. The Goedel Theorem and the Theory of Algorithm. Doklady Akad. Nauk SSSR 91 (1953), 737-740
6. On Computable Operations. Doklady Akad. Nauk SSSR 103 (1955), 773-776
7. Systems of Denumerable Sets and their Numbering. Doklady Akad. Nauk SSSR 105 (1955) 1155-1158
8. Computable Operations, Computable Operators, and Constructively Continuous Functions. Trudy 3-go Vsesoyunogo matem. s"yezda, Vol. I, Moscow (1956), 185
9. The Concept of the Program and Computable Operators. ibid. Vol. I, Moscow (1956), 186
10. On Algorithmic Reducibility. ibid. Vol. II, Moscow (1956), 66-69
12. Computable Operations and the Concept of the Program. Uspekhi Matematicheskikh Nauk 11:4 (70) (1956), 172-176
13. Contribution to the Theorem of Uniform Continuity. Uspekhi Matematicheskikh Nauk 12:1 (73) 99-142
14. A Few Comments on Denumerable Sets. Zeitschr. fur Math. Logik und Grundlagen der Mathematik. 3:2 (1957), 157-170

15. Concerning the Definition of the Case after A. N. Kolmogorov. Byull. ob"yedineniya po problemam mash. perevoda 5 (1957), 11-18
16. On the Definition of a Part of Speech in Set-Theoretical Language System. ibid. 5 (1957), 22-26

Vorob'yev, Nikolay Nikolayevich
4. Constructive Calculus of Predictions with Strong Negation Doklady Akad. Nauk SSSR 58 (1947), 1877-1879
5. The Problem of Derivability in the Constructive Calculus of Predictions with Strong Negation. ibid. 85 (1952), 689-692

Yablonskiy, Sergey Vsevolodovich
1. On Converging Sequences of Continuous Functions. M. Vestnik un-ta [Moscow, Herald of the University] 9 (1950), 13-29
3. On the Superposition of Functions of Algebraic Logic, Matematicheskiy sobrnik 30 (72) (1952), 329-348
5. Realization of a Linear Function in Class of π-Networks Doklady Akad. Nauk SSSR 94 (1954), 805-806
6. Functional Constructions in Multiple-Valued Logics. Trudy 3-go Vsesoyuznogo matem. s"yezda, Vol. II, Moscow (1956), 71-73
7. On Functional Completeness in Three-Valued Calculus. Doklady Akad. Nauk SSSR 95 (1954), 1153-1155
9. On One Family of Classes of Functions of Algebraic Logic, Admitting of a Simple Network Realization. Trudy 3-go Vsesoyuznogo matem. s"yezda, Vol. II, Moscow (1956), 149
10. On Tests for Electric Networks. Uspekhi Matematicheskikh Nauk 10:4 (66) (1956), 182-194 (together with I. S. Chegis)
11. On Classes of Functions of Algebraic Logics, Admitting of Simple Circuit Realization. Uspekhi Matematicheskikh Nauk 12:6 (78) (1957), 189-196

Yanovskaya, Sof'ya Aleksandrovna
29. From the History of the Axiomatic Method. Trudy 3-go Vsesoyuznogo matem. s"yezda, Vol. II, Moscow (1956), 105

Yesenin-Vol'pin, Aleksandr Sergeyevich
3. Non-Provability of the Suslin Hypothesis without the Aid of the Axiom of Choice in the System of Bernays-Mostowski Axioms. Doklady Akad. Nauk SSSR (1954), 9-12
4. Proof of Non-Contradiction of Classical Arithmetic with the Aid of Induction to E_0 (After Schutte). Appendix VII to the book by S. C. Kleene "Introduction to Meta-Methematics" (Rus. Transl.) Moscow (1957)

Zaslavskiy, Igor' Dmitrievich
3. Refutation of Certain Theorems of Classical Analysis in Constructive Analysis. Uspekhi Matematicheskikh Nauk 10:4 (66) (1955), 209-210

5. Certain Features of Constructive Functions of Real Variables Compared with the Classical Ones. Trudy 3-go Vsesoyuznogo matem. s"yezda [Works of the Third All-Union Mathematical Congress] Vol. I, Moscow (1958), 181-182
6. On Constructive Dedekind Cuts, ibid. Vol. I. Moscow (1956), 182-183

Zynov'yev, A. A.
1. Broadening the Topics of Logical Research; on the Works of the Seminar on Logic at the Institute of Philosophy Academy of Sciences. Voprosy Filozofii [Problems of Philosophy] 3, (1957), 211-215

Zykov, Aleksandr Aleksandrovich
2. Problem of Spectrum in the Broadened Calculus of Predicates. Izvesitya Akad. Nauk SSSR, ser. mat. 17 (1955), 63-76

Zhegalkin, Ivan Ivanovich
2. Arithmetization of Symbolic Logic. Matematicheskiy sbornik [Mathematics Collection], 35, (1928), 311-378
8. On the Problem of Solvability, ibid. 6 (48) (1939), 185-198

Zhyravlev, Yuriy Ivanovich
2. On the Separability of Subsets of the Vertices of an n-Dimensional Unit Cube, Doklady Akad. Nauk SSSR, 113 (1957), 264-267

END